Feynman rules

- tree level
- loops (→ ghosts)

loop integrals → ∞ UV

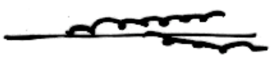
⇒ absorb ∞ in α_s, m, Z

⇒ $\alpha_s(q^2)$

$$\alpha_s(q^2)_{\overline{MS}} = \frac{12\pi}{33-2n_f}\,\frac{1}{\log\frac{-q^2}{\Lambda^2_{QCD}}}$$

- dependence on renormalisation
  - scale
  - scheme

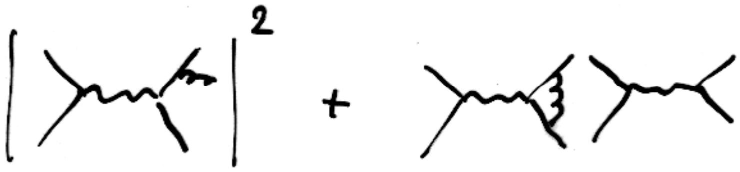
bremstrahlung → ∞ IR

collinear :

soft :

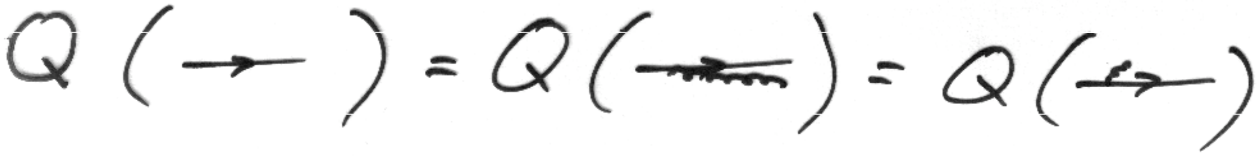
large distances

⇒ force short distances:

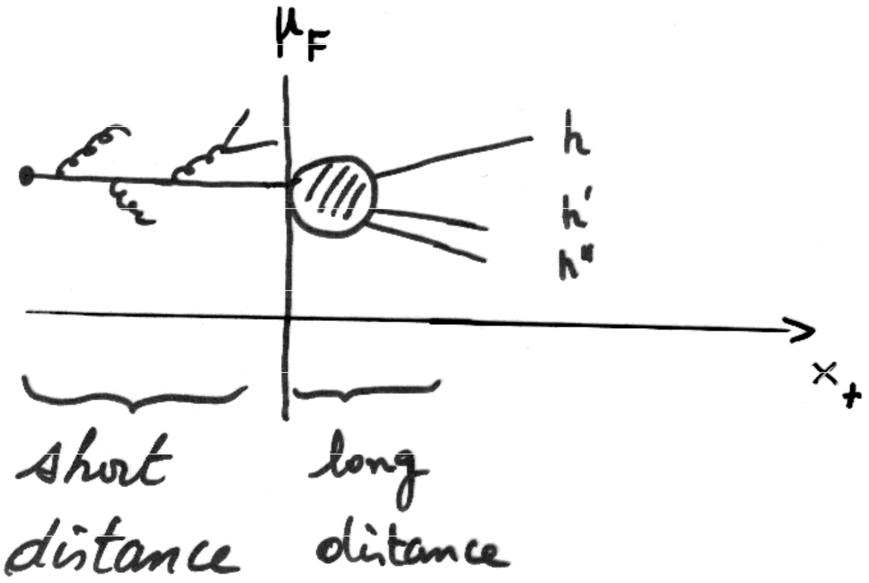
- inclusive quantities


finite

Other I.R. solutions:

a) 

⇒ Q is IR stable

b) 

Other I.R. stable quantities:

i^{th} hadron : $E_i\,,\ \theta_i\,,\ \phi_i$

$$\frac{d\sigma^{[N]}}{d\Omega_2\, dE_2 \dots d\Omega_N\, dE_N}$$

$$I = \frac{1}{2}\int d\Omega_2\ \frac{d\sigma^{[2]}}{d\Omega_2}\ w_2(p_1^\mu, p_2^\nu)$$

$$+\ \frac{1}{3!}\int d\Omega_2\, dE_3\, d\Omega_3\ \frac{d\sigma^{[3]}}{d\Omega_2\, dE_3\, d\Omega_3}\ w_3(p_1, p_2, p_3)$$

$$+\ \frac{1}{4!}\int d\Omega_2\, dE_3\, d\Omega_3\, dE_4\, d\Omega_4$$

$$\frac{d\sigma^{[4]}}{d\Omega_2\, dE_3\, d\Omega_3\, dE_4\, d\Omega_4}\ w_4(p_1, p_2, p_3, p_4)$$

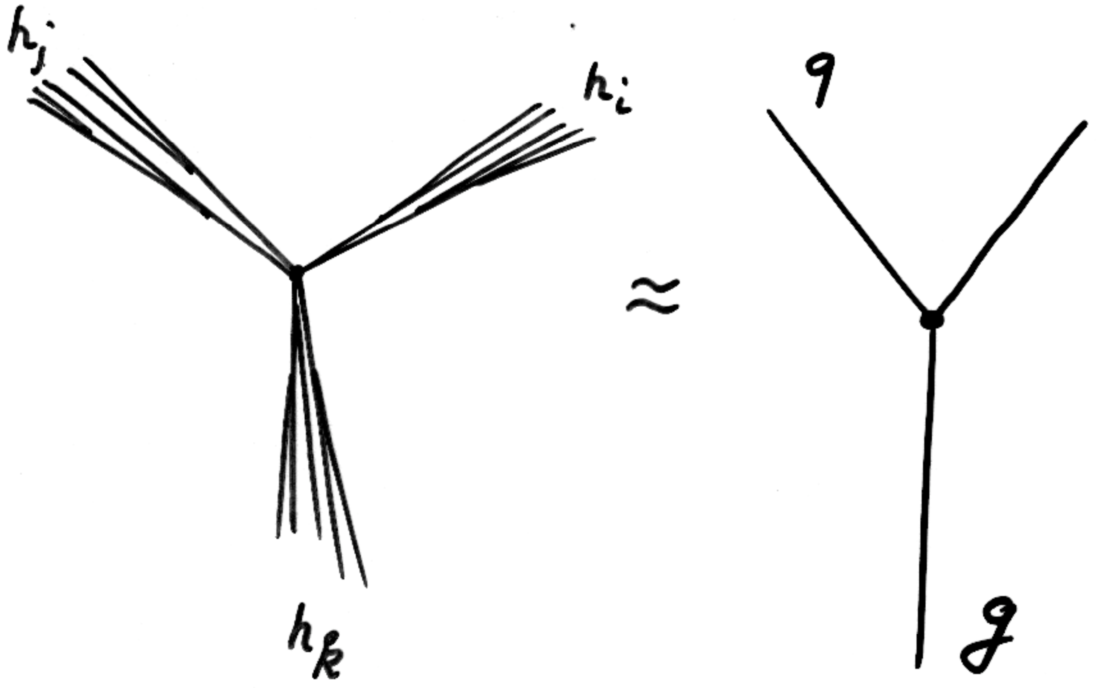
$$+ \dots$$

I infrared safe if

$$w_{n+1}(p_1 \dots, \underbrace{(1-\lambda)p_n, \lambda p_n}_{\text{bremstrahlung}}) = w_n(p_1 \dots \overset{\uparrow}{p_n})$$

virtual

: same I



(far from IR or collinear)

Examples:

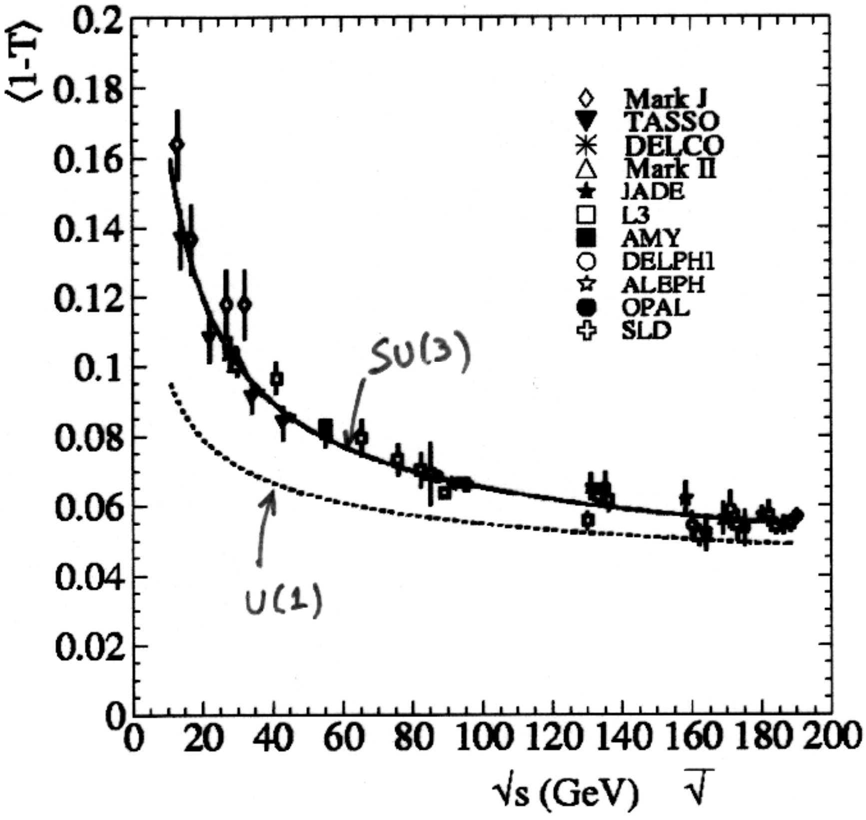
. total σ : $w_j = 1$

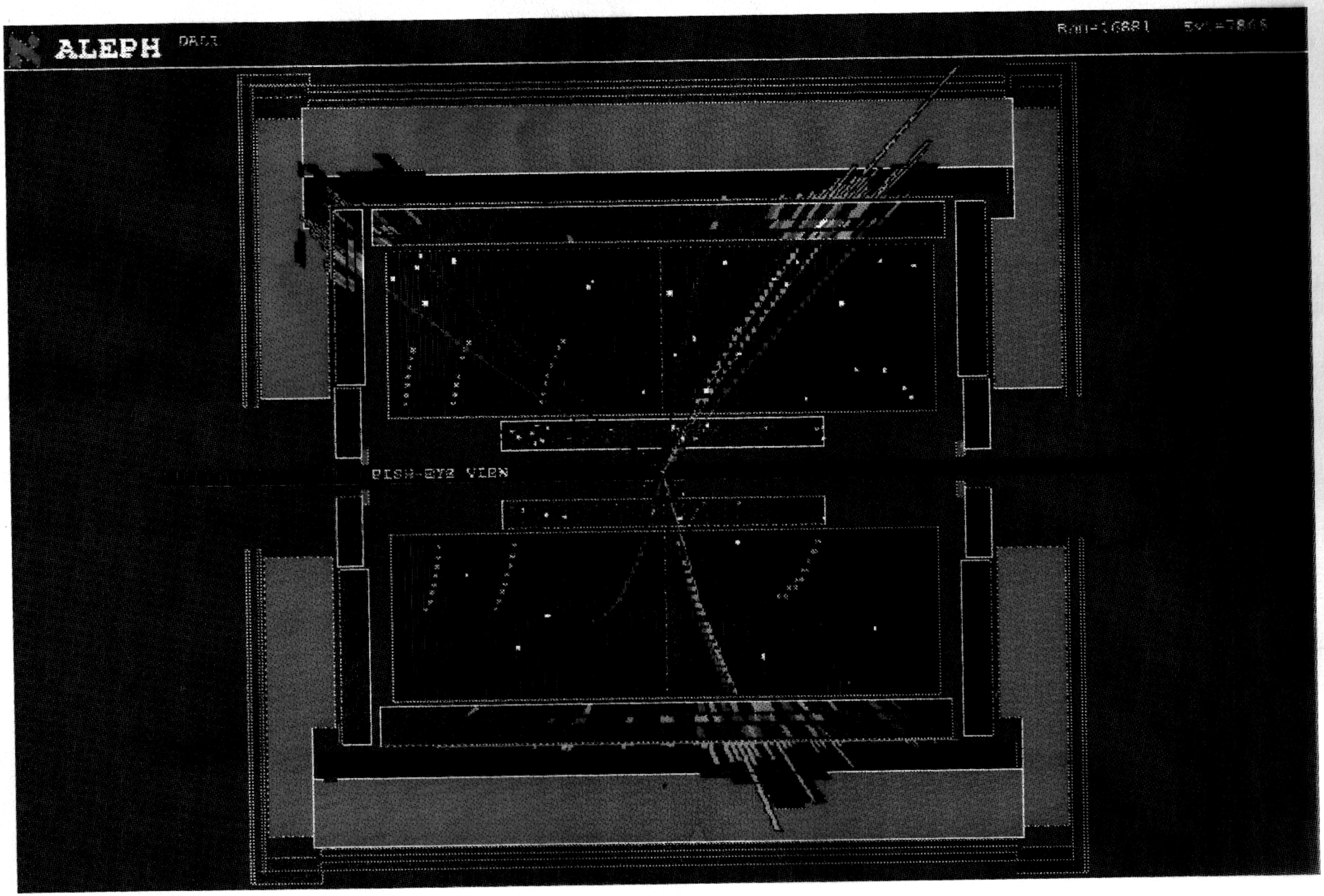
. thrust : $\tau_n(p_1 \ldots p_n) = \max\limits_{\vec{u}} \frac{\sum |\vec{p}_i \cdot \vec{u}_i|}{\sum |\vec{p}_i|}$

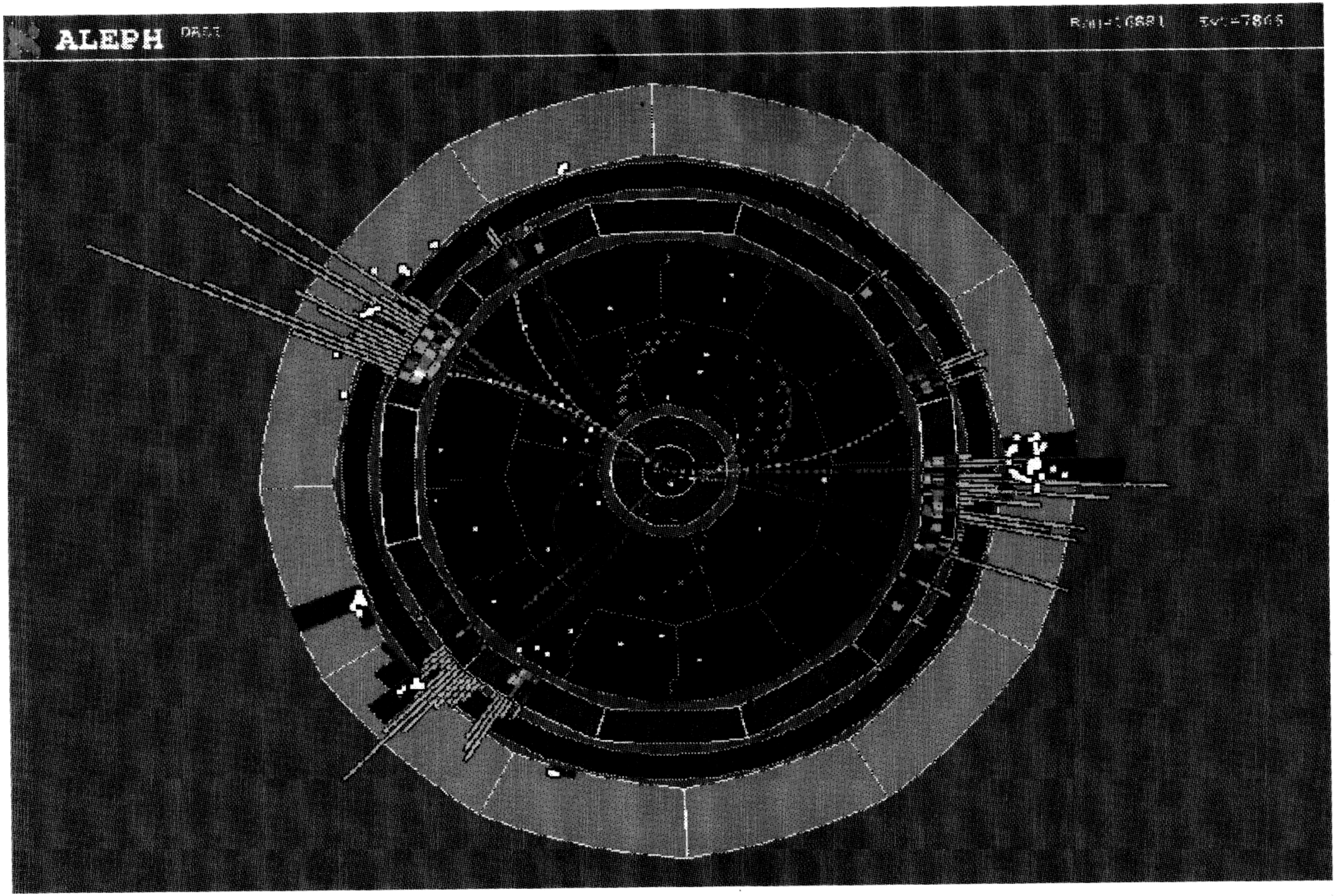
$$w_n = \frac{\delta(\tau - \tau_n)}{\sigma_{tot}}$$

. E-E correlations :

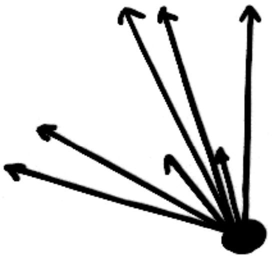
$$w_n = \sum_{ij} \frac{E_i E_j}{s} \delta(\cos\theta_{ij} - \cos\theta)$$


⟨1-T⟩
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
20
40
60
80
100
120
140
160
180
200
√s (GeV)
√
◇ Mark J
▼ TASSO
✳ DELCO
△ Mark II
★ JADE
□ L3
■ AMY
○ DELPHI
☆ ALEPH
● OPAL
✣ SLD
SU(3)
U(1)


ALEPH
DALI
Run=16881
Evt=7865
FISH-EYE VIEW

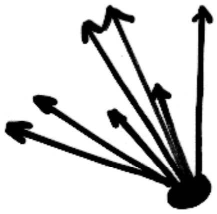

ALEPH
DALI
Run=10881 Evt=7865

Jet definition



p_1
p_2
$\vdots$
p_n

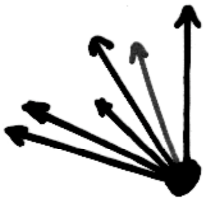
→ find i,j | $(p_i+p_j)^2$ is min



- if $(p_i+p_j)^2 > m_{cut}^2$ → STOP
- else $(p_i+p_j^2)^2 < m_{cut}^2$

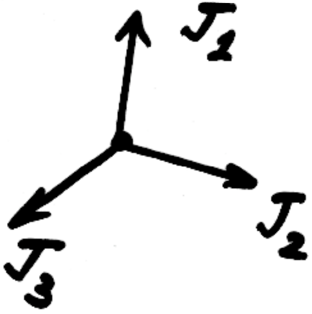
replace p_i+p_j by $p_k = p_i+p_j$

p_1
$\vdots$
p_k
$\vdots$
p_{n-1}



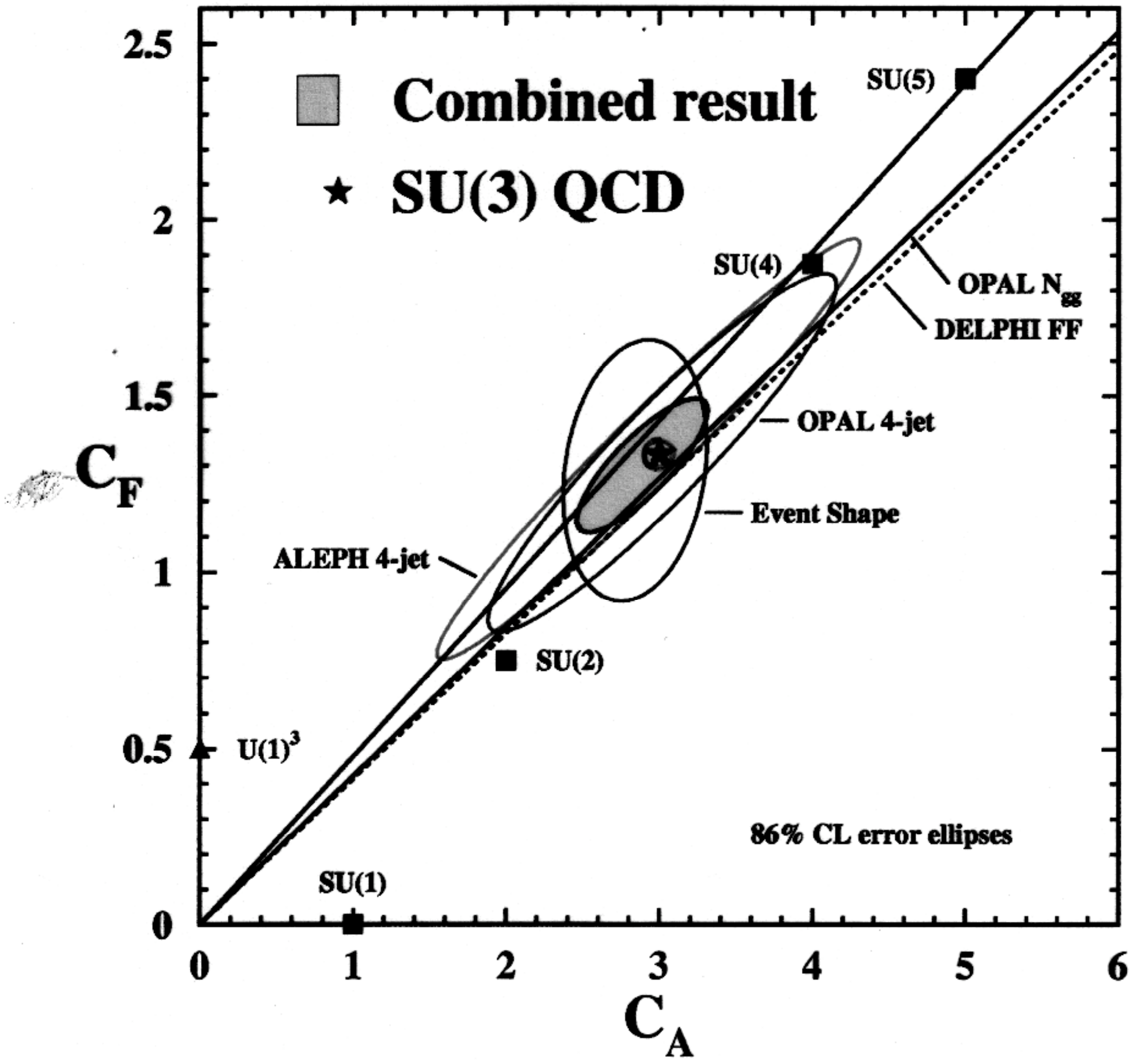
goto pairing

=> final state = clusters

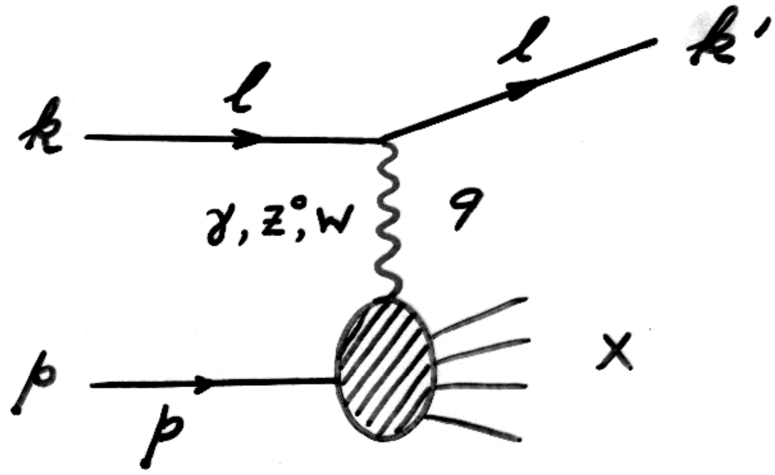


$(p_i+p_j)^2 > m_{cut}^2$

(= Jets)


Combined result
★ SU(3) QCD
SU(5)
SU(4)
OPAL N_{gg}
DELPHI FF
OPAL 4-jet
Event Shape
ALEPH 4-jet
SU(2)
$U(1)^3$
SU(1)
86% CL error ellipses
C_F
C_A
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5
6

Deeply inelastic scattering



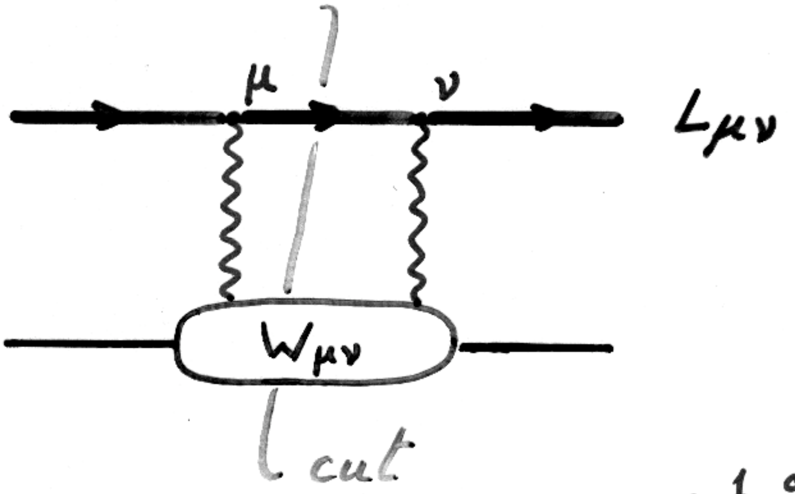
$$W^2 = (p+q)^2 = M_x^2 \gg m_p^2$$

$$Q^2 = -q^2 \gg m_p^2$$

$$x_{Bj} = \frac{Q^2}{2p\cdot q} \qquad p\cdot q = \nu = m_p E_\gamma$$

$$y = \frac{p\cdot q}{p\cdot k} = \frac{E_\gamma}{E_e}$$

$d\sigma \sim$



$= |\mathcal{M}|^2$

$$d\sigma = \frac{4\alpha^2}{s}\,\frac{d^3k'}{2k'_0}\,\frac{\overset{1 \text{ or } \frac{1}{64\sin^4\theta_W}}{K}}{\underset{\text{or } (q^2-M_V^2)^2}{(q^2)^2}}\,L^{\mu\nu}W_{\mu\nu}$$

$$L_{\mu\nu} = \frac{1}{2} Tr\left(k\cdot\gamma \, \Gamma^{\mu} k'\cdot\gamma \, \Gamma^{\nu} \right)$$

$\gamma^{\mu} \nearrow$ (to Γ^{μ})

γ^{μ} $\quad ep \rightarrow ex$

$\gamma^{\mu}(1-\gamma_5)$ $\quad \nu p \rightarrow ex$

$\gamma^{\mu}(1+\gamma_5)$ $\quad \bar{\nu} p \rightarrow ex$

$$W_{\mu\nu} \quad : \quad \begin{cases} \text{tensor} \\ W^{\dagger}_{\mu\nu} = W_{\mu\nu} \\ q^{\mu} W_{\mu\nu} \quad = 0 \quad \text{for } \gamma_{\mu} \\ \qquad\qquad \simeq 0 \quad \text{for } \gamma_5 \gamma_{\mu} \end{cases}$$

$$\begin{aligned} W_{\mu\nu} = & -\left(g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) F_1(x, Q^2) \\ & + \frac{1}{p\cdot q} \left(p_{\mu} - q_{\mu} \frac{p\cdot q}{q^2} \right) \left(p_{\nu} - q_{\nu} \frac{p\cdot q}{q^2} \right) F_2(x, Q^2) \\ & - \frac{i}{p\cdot q} \, \epsilon_{\mu\nu\lambda\sigma} \, p^{\lambda} q^{\sigma} \, F_3(x, Q^2) \end{aligned}$$

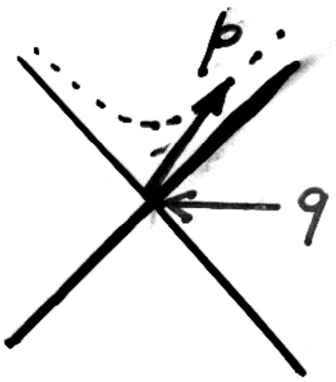
$$\frac{d\sigma}{dx\,dy} = n(Q^2) \left[y F_1 + \frac{1-y}{xy} F_2 + \delta_{\nu} \left(1 - \frac{y}{2} \right) F_3 \right]$$

$n = \dfrac{4\pi\alpha^2}{Q^2}$ $\qquad \delta_{\nu} = 0$ $\qquad ep$

$n = \dfrac{\pi \alpha^2 Q^2}{4 \sin^4\theta_W \left(Q^2 + M_W^2 \right)^2}$ $\qquad \delta_{\nu} = 1$ $\qquad \nu p$

$n =$ $\qquad$ " $\qquad \delta_{\nu} = -1$ $\qquad \bar{\nu} p$

Space - time picture :



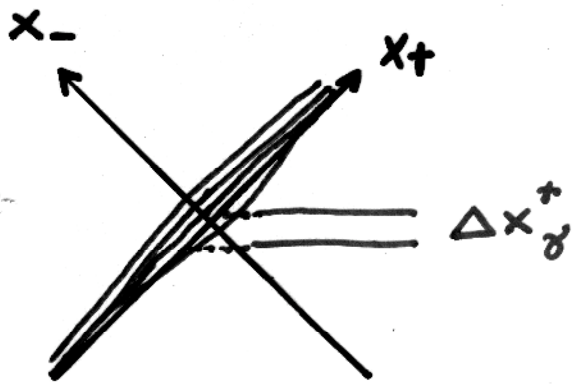
$$-q_+ = q_- = \frac{Q}{\sqrt{2}} \qquad q_T = 0$$

$$p_T = 0$$

$$\frac{1}{\sqrt{2}} Q (p_+ - p_-) = \nu = \frac{Q^2}{2x}$$

$$2 p_+ p_- = m_p^2$$

choose $p_+ \gg p_-$ $\Rightarrow \left\{ \begin{array}{l} p_+ \simeq \dfrac{Q}{\sqrt{2}\, x} \\ p_- \simeq \dfrac{m^2 x}{\sqrt{2}\, Q} \end{array} \right.$



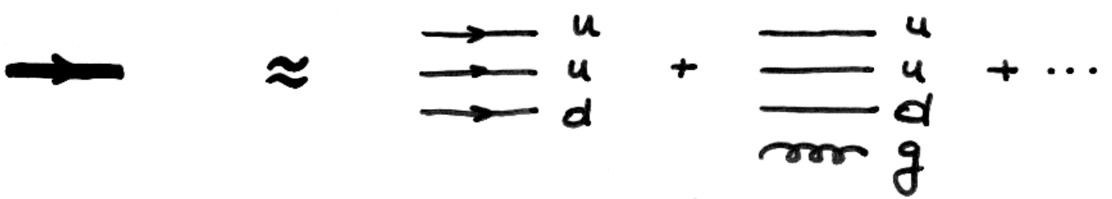
x_+ spread

x_- small

$$\Delta x^+ \simeq \underbrace{\frac{1}{m}}_{\text{size}} \times \underbrace{\frac{Q}{m}}_{\text{boost}}$$

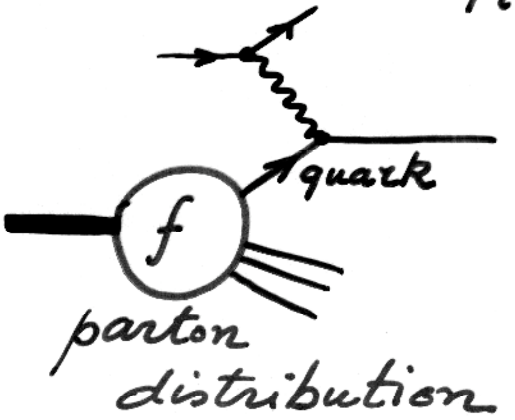
$$\Delta x^+_\gamma \approx \frac{1}{Q} \lll \Delta_p x^+ \quad (x_{Bj} \sim 1)$$

The interactions between quarks are "frozen"



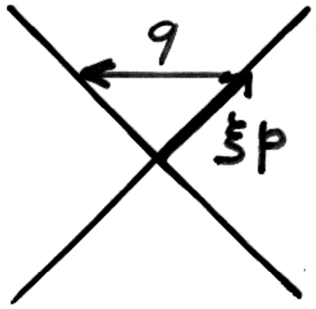
$$\xi_i \equiv \frac{p_i^+}{p^+} \qquad 0 < \xi_i < 1$$

$$p_i^- \approx 0$$



$$d\sigma = \int_0^1 d\xi \sum_a f_{a/p}\, d\hat{\sigma}$$

$$+ \, O\left(\frac{m}{Q}\right)$$



final quark:

$\xi p + q$ on − axis

$$\Rightarrow \xi p^+ + q^+ = 0$$

$$\xi \frac{Q}{x\sqrt{2}} + \left(-\frac{Q}{\sqrt{2}}\right) = 0$$

$$\Rightarrow \xi = x$$

spin 1/2

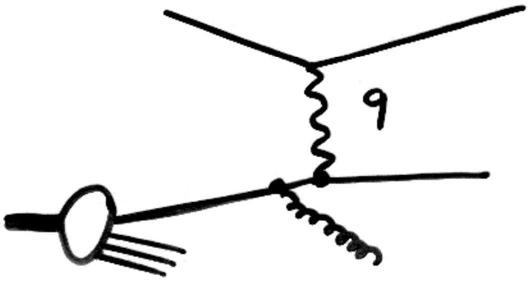
$$2x F_1^{\gamma p} = F_2^{\gamma p} = \sum_{q,\bar{q}} e_q^2\, q(x)$$

$$= x \left[\frac{4}{9} \left(u(x) + \bar{u}(x) + c(x) + \bar{c}(x) \right) \right.$$

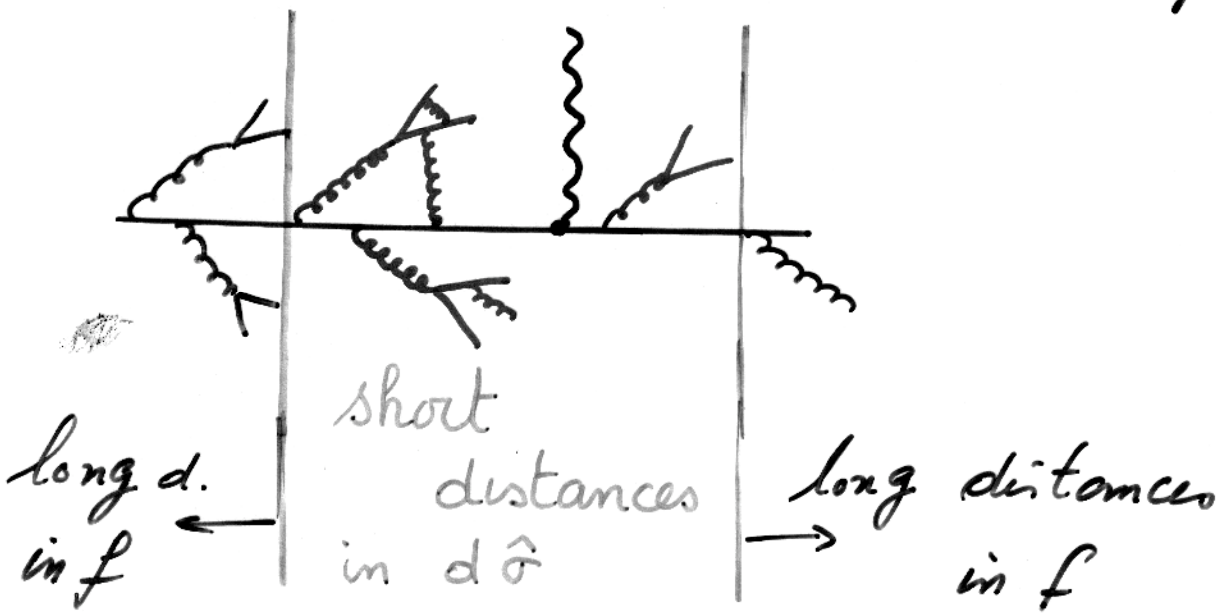
$$\left. + \; \frac{1}{9} \left(d(x) + \bar{d}(x) + s(x) + \bar{s}(x) \right) \right]$$

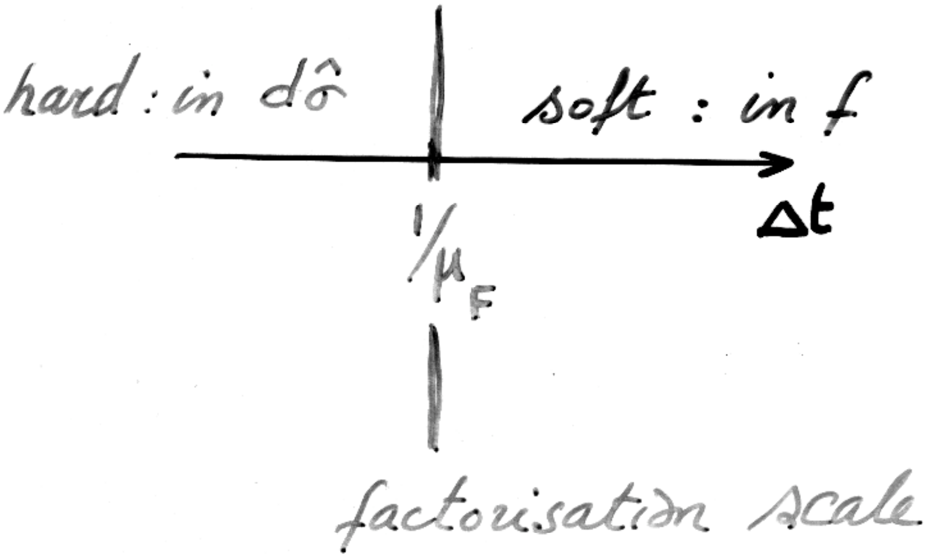
Gribov - Lipatov (DGLAP)

fluctuations $\Delta x_p^+ \approx \frac{Q}{m^2}$
what about short ones?

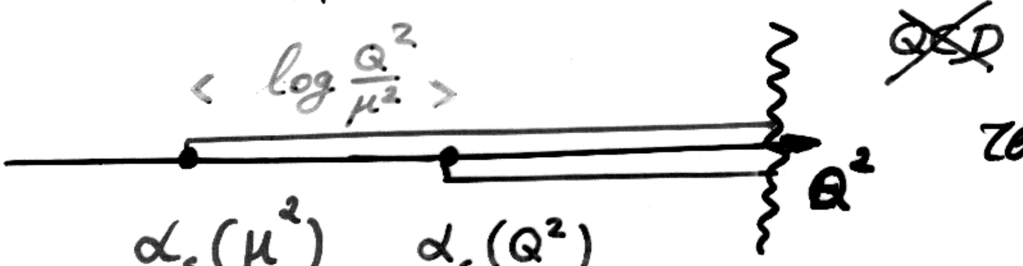


$\Rightarrow$ corrections $\sim \log \frac{Q^2}{\mu_{IR}^2}$





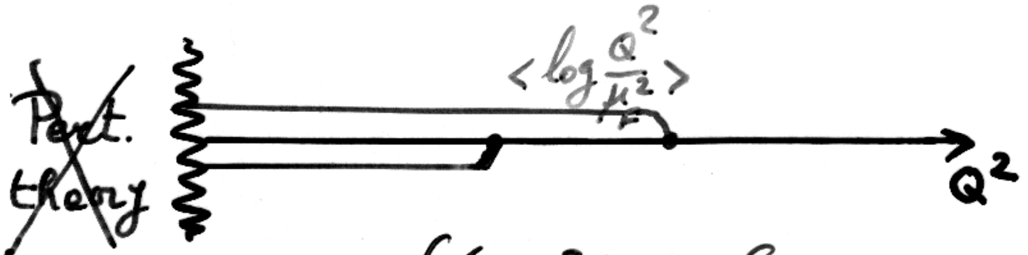
Change $\begin{cases} \mu & \Rightarrow \text{R.G.} \\ Q^2/\mu_F^2 & \Rightarrow \text{evolution} \end{cases}$



renormalise

$\alpha_s(\mu^2)$ $\alpha_s(Q^2)$

$\log\left(\frac{\mu^2}{\infty}\right)$ $\log\left(\frac{Q^2}{\infty}\right)$



factorise

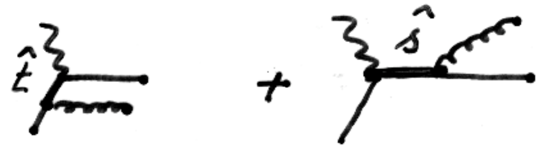
$f(x,\mu_F^2)$ $f(x,Q^2)$

$\log\left(\frac{\mu_F^2}{\infty}\right)$ $\log\left(\frac{Q^2}{\infty}\right)$

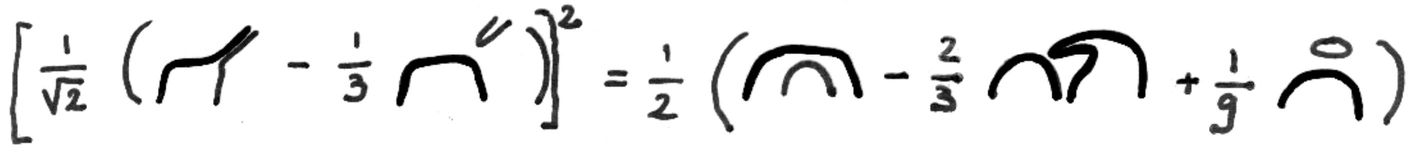
$$\frac{\partial}{\partial \log Q^2} f_{a/p}(x,Q^2) = \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}\left(\frac{x}{\xi}\right) f_{b/p}(\xi,Q^2)$$

evolution equation

Example : P_{qq}

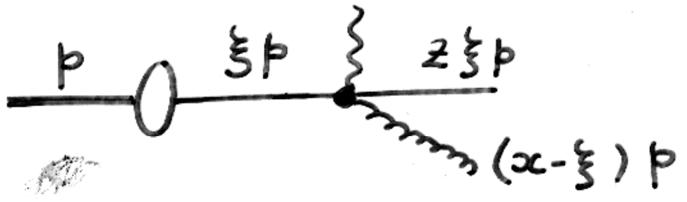


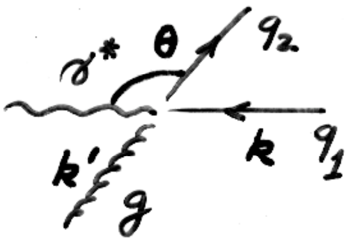
$$|\overline{\mathcal{M}}|^2_{\text{Compton}} = 32\pi^2\alpha^2\left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}\right) \times C_F\frac{\alpha_s}{\alpha}e_q^2$$



$$= \frac{1}{2}\left(9 - 2 + 1\right) = 4$$

average over initial colour $\Rightarrow \frac{4}{3} = C_F$





$$\hat{t} = -2kk'(1-\cos\theta)$$
$$\hat{u} = -2kk'(1+\cos\theta)$$
$$\hat{s} = 4k'^2$$

collinear:

$\theta \approx 0$

$\hat{t} \to 0 \qquad \hat{u} \simeq -\hat{s} - Q^2$

$$4kk' = \hat{s} + Q^2$$

$$p_T = k'\sin\theta \qquad dp_T = k'\cos\theta\, d\theta \simeq k'd\theta$$

$$d\Omega = 2\pi\sin\theta\, d\theta = \frac{2\pi}{k'^2}p_T\, dp_T = \frac{4\pi}{\hat{s}}dp_T^2$$

$$\Rightarrow \frac{d\sigma}{dp_T^2} = \frac{1}{16\pi\hat{s}^2}|\mathcal{M}|^2$$

$$\approx \frac{8\pi e_q^2\alpha\alpha_s}{3\hat{s}^2}\left(\frac{1}{-\hat{t}}\right)\left(\hat{s} + \frac{2(\hat{s}+Q^2)Q^2}{\hat{s}}\right)$$

$$z = \frac{Q^2}{2\xi p\cdot q} = \frac{Q^2}{(\xi p+q)^2-q^2} = \frac{Q^2}{\hat{s}+Q^2}$$

$$\Rightarrow \frac{d\sigma}{dp_T^2} = \underbrace{\frac{4\pi^2\alpha e_i^2}{\hat{s}}}_{\substack{\text{lowest}\\ \text{order}\\ \hat{\sigma}_0}} \underbrace{\frac{1}{p_T^2}}_{\substack{\text{collinear}\\ \text{div.}}} \frac{\alpha_s}{2\pi} \underbrace{\frac{4}{3}\left(\frac{1+z^2}{1-z}\right)}_{\substack{\text{splitting}\\ P_{qq}(z)}}$$

$$\hat{\sigma}(z) = \hat{\sigma}_0 \cdot \log\frac{Q^2}{\mu_F^2} \cdot P_{qq}(z)$$

$$f_{q/p}(x,Q^2)$$

$$= q(x,Q^2) = q_0(x) + \int_0^1 d\xi\, dz\, \delta(\xi z - x)\, q(\xi,Q^2)\, \frac{\hat{\sigma}}{\hat{\sigma}_0}$$

$$= q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\, q_0(\xi) \quad P_{qq}\left(\frac{x}{\xi}\right) \log\frac{Q^2}{\mu^2}$$

true if $q_0(x) \rightarrow q(x,Q^2)$

$\Rightarrow$ evolution:

$$\frac{\partial}{\partial \log Q^2} q(x,Q^2) = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{d\xi}{\xi} P\left(\frac{x}{\xi}\right) q(x,Q^2)$$

Complications:

virtual diagrams:

$= c\,\delta(1-z)$ regularises $z \to 1$

conservation of quark (+ antiquark)

$$\Rightarrow \underbrace{\int \left(P_{qq} + c\,\delta(1-z)\right)}_{\text{probability to change } n_q} dz = 0$$

Replace P by P_+: $P \sim \frac{P_0}{1-z}$

$$\int_0^1 \frac{f(z)}{(1-z)_+}\, dz \equiv \int_0^1 \frac{f(z)-f(1)}{1-z}\, dz$$

« + prescription »

Other branchings:

P_{gg}

P_{qg}

....

The evolution equations:

$$\underset{\substack{\uparrow \\ \log Q^2 \\ \text{resummed}}}{\frac{Q^2\partial}{\partial Q^2}} \underset{\substack{\uparrow \\ q = \left\{\substack{\text{quark} \\ \text{antiquark}}\right. \\ n_f \text{ flavours}}}{\begin{pmatrix} q_i(x,Q^2) \\ g(x,Q^2) \end{pmatrix}} = \frac{\alpha_s(Q^2)}{2\pi} \sum_{q_j \bar{q}_j} \int_{\underset{\substack{\nearrow \\ \xi z = x}}{x}}^{1} \frac{d\xi}{\xi} \begin{pmatrix} P_{q_i q_j}\left(\frac{x}{\xi}\right) & P_{q_i g}\left(\frac{x}{\xi}\right) \\ \underset{\substack{\uparrow \\ \text{mixed} \\ \text{evolution}}}{P_{g q_j}\left(\frac{x}{\xi}\right)} & P_{gg}\left(\frac{x}{\xi}\right) \end{pmatrix} \begin{pmatrix} q(\xi,Q^2) \\ g(\xi,Q^2) \end{pmatrix},$$

lowest order:

$$\left\{\begin{aligned} P_{qq}^{(x)} &= C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\,\delta(1-x)\right] \qquad C_F = \frac{4}{3} \\ P_{qg}^{(x)} &= T_R\left[x^2 + (1-x)^2\right] \qquad T_R = \frac{1}{2} \\ P_{gq}(x) &= C_F\left[\frac{1+(1-x)^2}{x}\right] \\ P_{gg}(x) &= 2C_A\left[\frac{x}{(1-x)_+} + x(1-x) + \frac{1-x}{x}\right] + \delta(1-x)\,\frac{11C_A - 4n_fT_R}{6} \end{aligned}\right.$$

$$P_{q_i \neq q_j} = 0 \qquad (C_A = 3)$$

note singularity at $x = 0$ ⟶ $P_{gq}(x)$, $P_{gg}(x)$

Strategy:

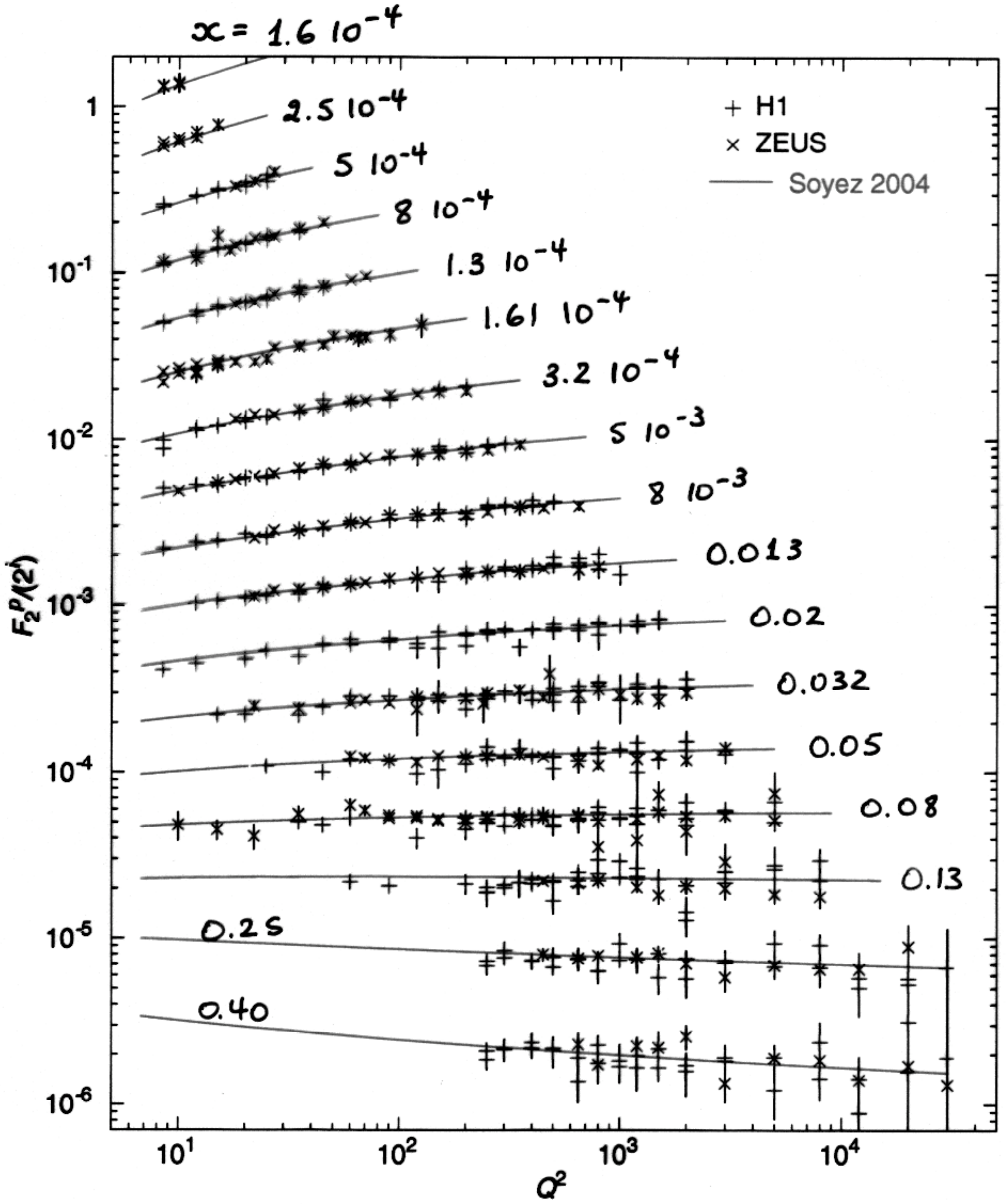
$$F_2 = x \sum e_q^2 \, q(x, Q^2)$$

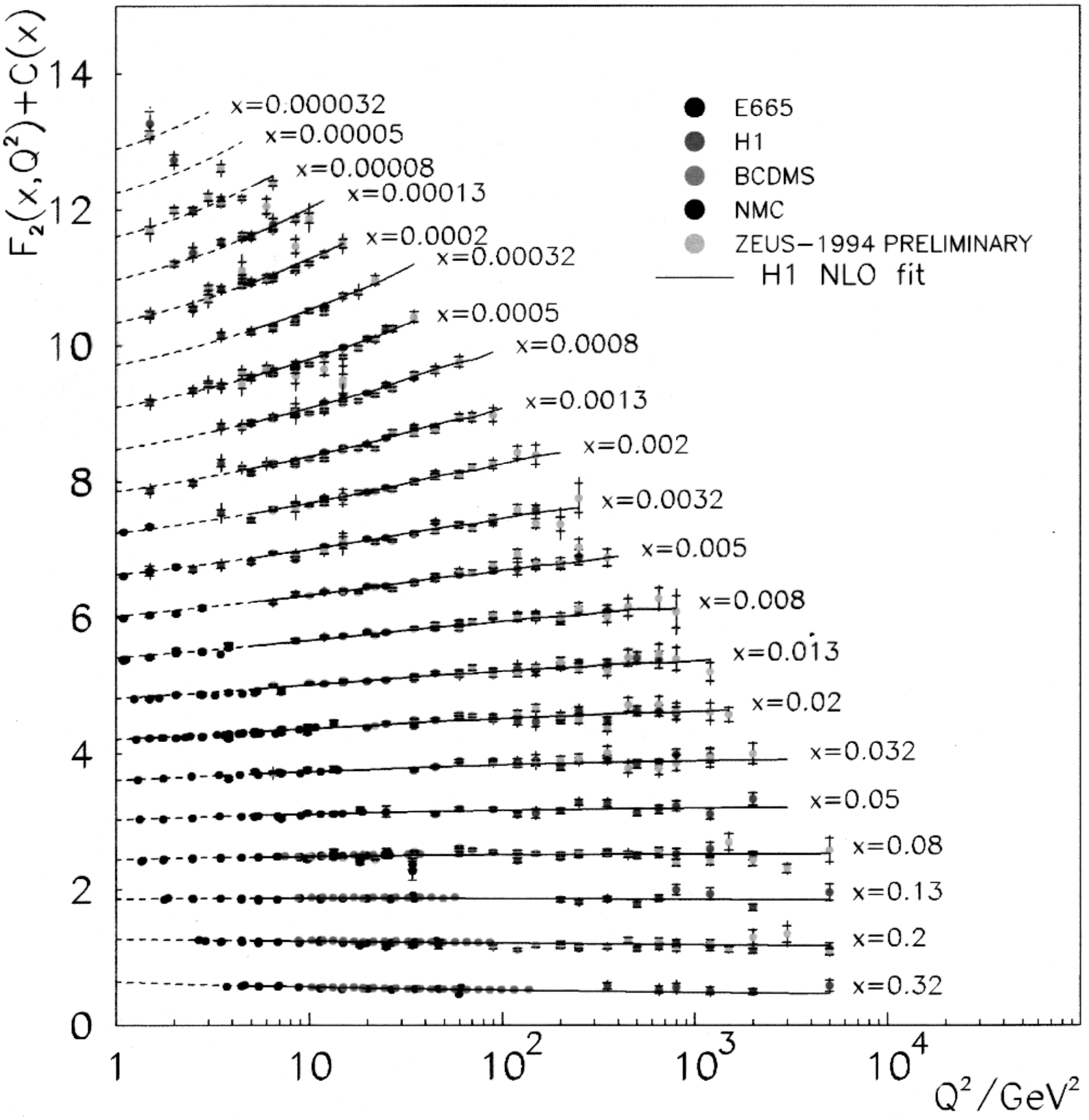
choose $\begin{cases} q(x, Q_0^2) \\ g(x, Q_0^2) \end{cases}$ $Q_0 = \mu$

evolve + fit

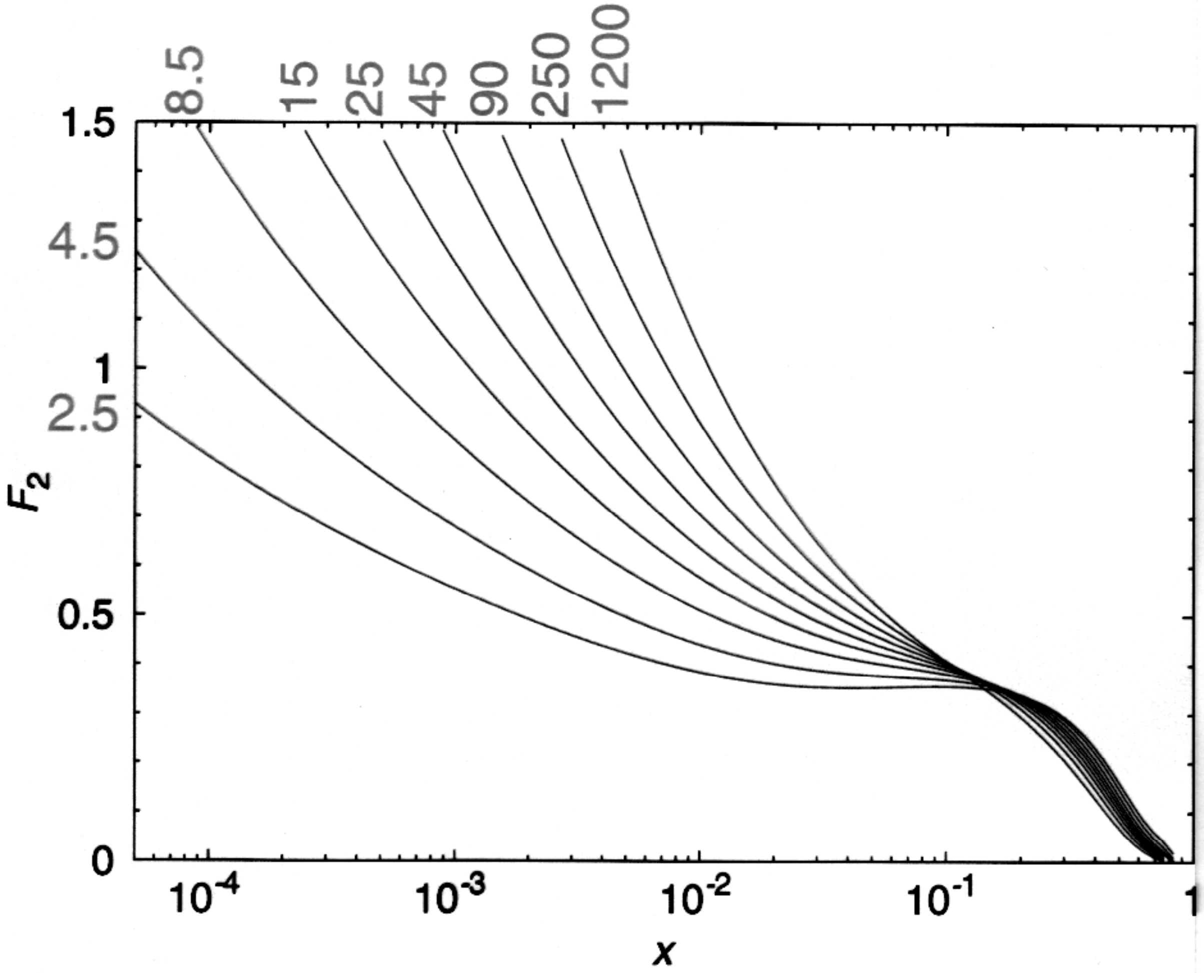
$\Rightarrow \begin{cases} q(x, Q^2) \\ g(x, Q^2) \end{cases}$

$$\left.\begin{array}{l} \bullet\ \text{GRV} \\ \bullet\ \text{MRST} \\ \bullet\ \text{CTEQ} \\ \bullet\ \text{Soyez} \\ \bullet\ \text{H1, ZEUS} \end{array}\right\} \text{GLOBAL FITS}$$

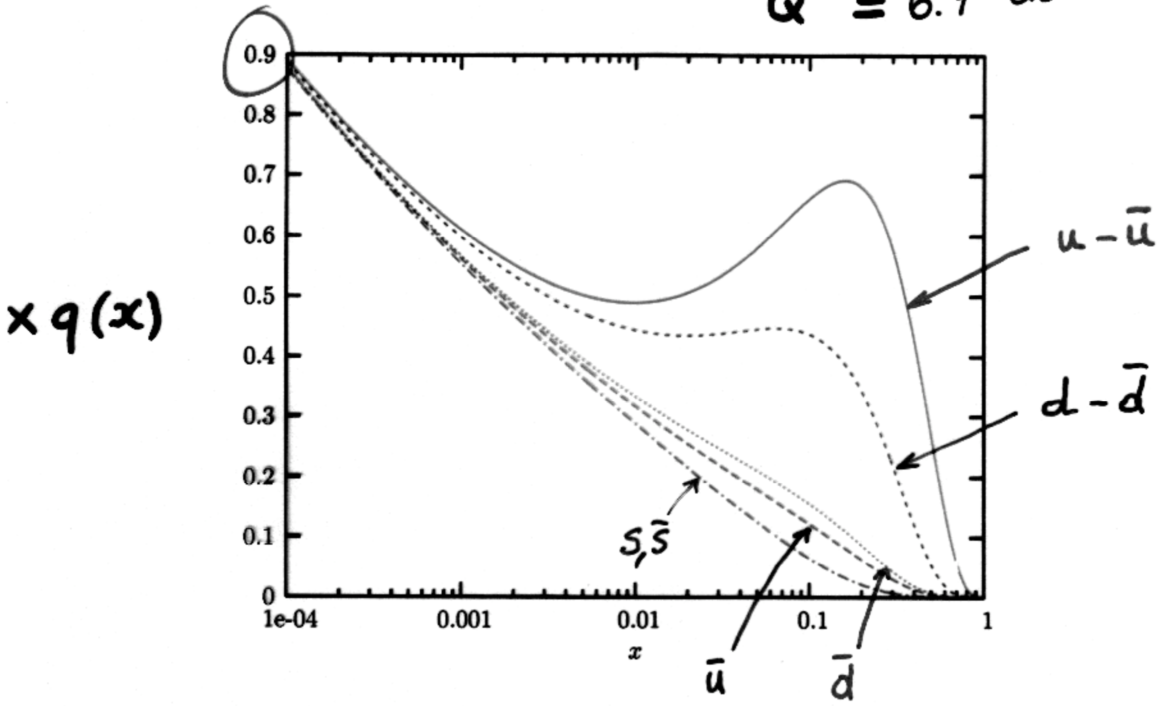

x = 1.6 10^-4
2.5 10^-4
5 10^-4
8 10^-4
1.3 10^-4
1.61 10^-4
3.2 10^-4
5 10^-3
8 10^-3
0.013
0.02
0.032
0.05
0.08
0.13
0.25
0.40
+ H1
× ZEUS
Soyez 2004
1
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
F_2^p/(2^i)
10^1
10^2
10^3
10^4
Q^2


$F_2(x,Q^2)+C(x)$
14
12
10
8
6
4
2
0
1
10
10^2
10^3
10^4
Q^2/GeV^2
x=0.000032
x=0.00005
x=0.00008
x=0.00013
x=0.0002
x=0.00032
x=0.0005
x=0.0008
x=0.0013
x=0.002
x=0.0032
x=0.005
x=0.008
x=0.013
x=0.02
x=0.032
x=0.05
x=0.08
x=0.13
x=0.2
x=0.32
● E665
● H1
● BCDMS
● NMC
● ZEUS-1994 PRELIMINARY
H1 NLO fit

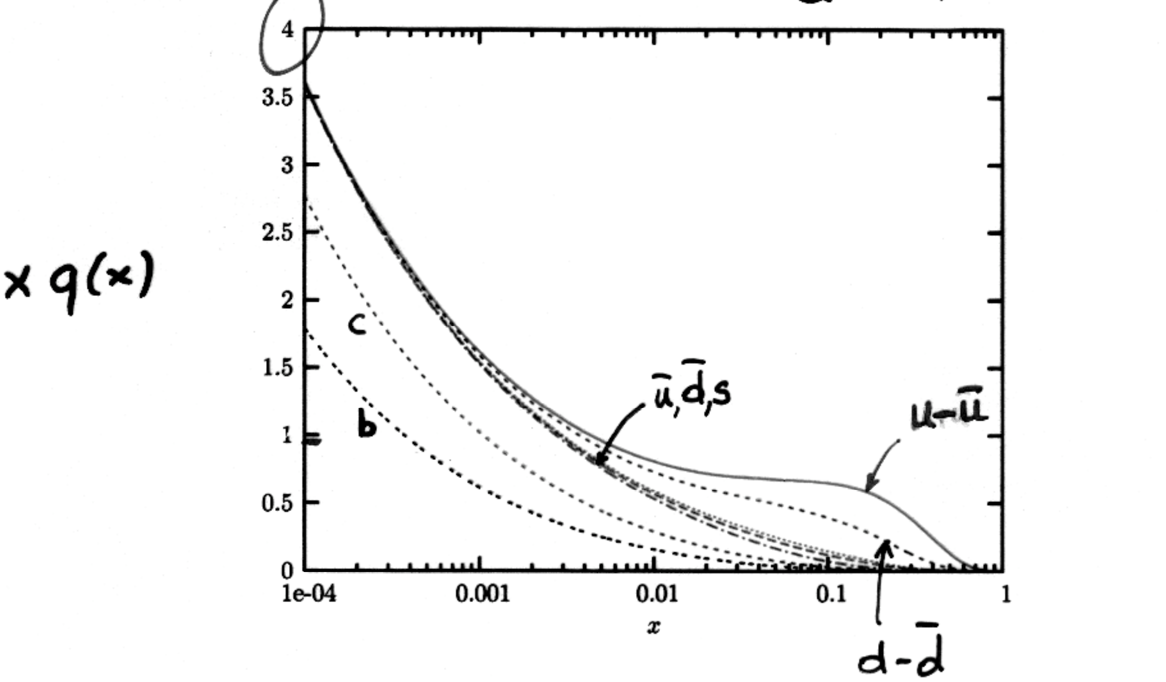
Evolution of F_2 :

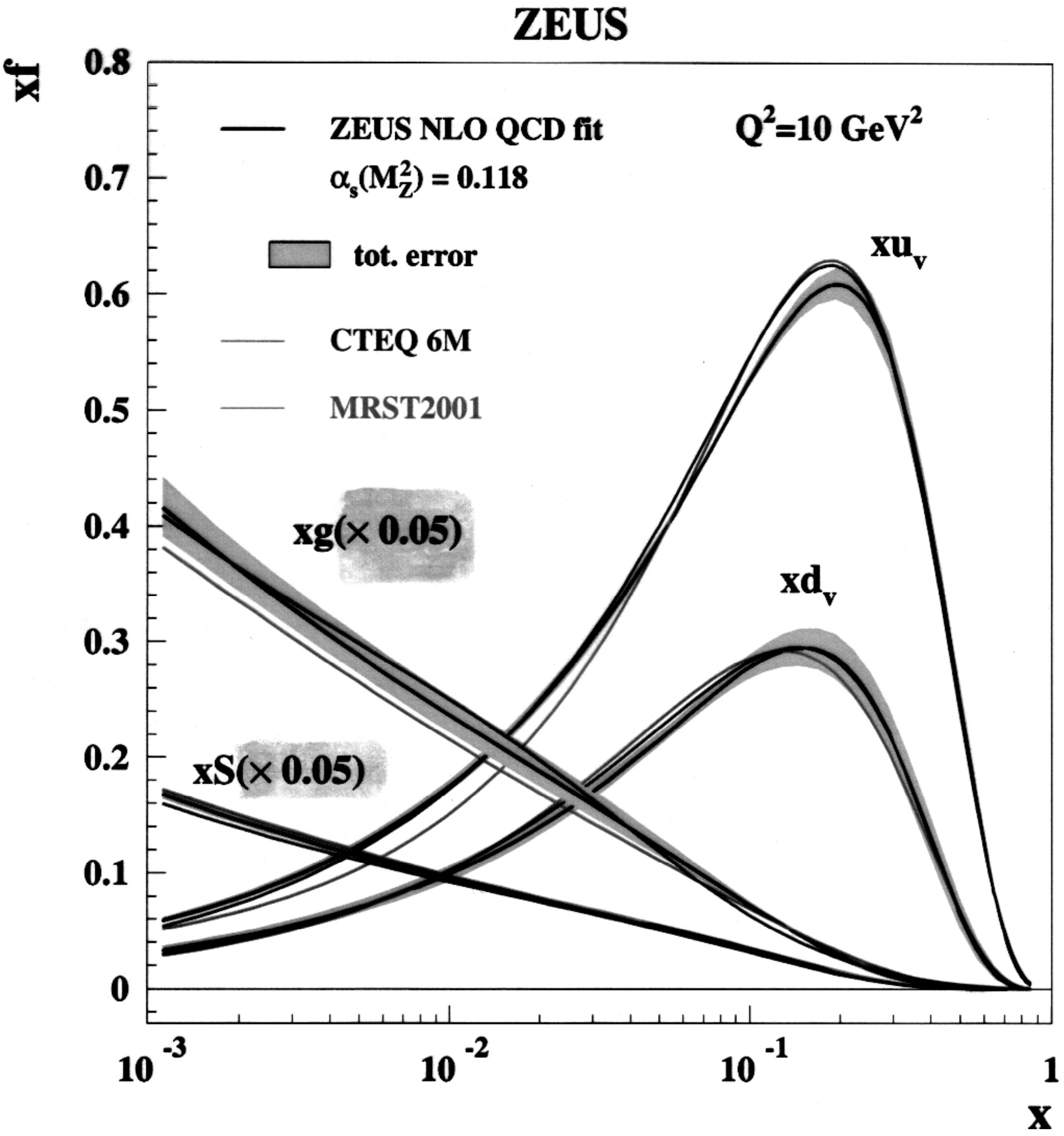


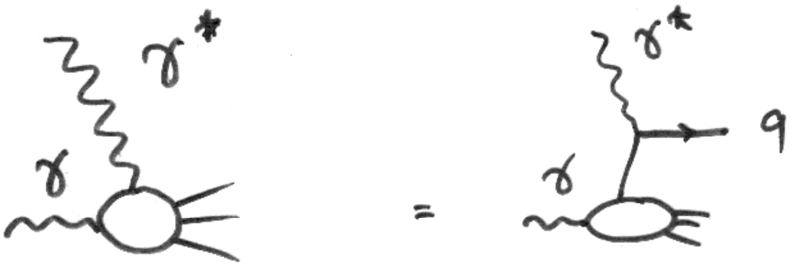
$Q^2 = 6.7$ GeV^2

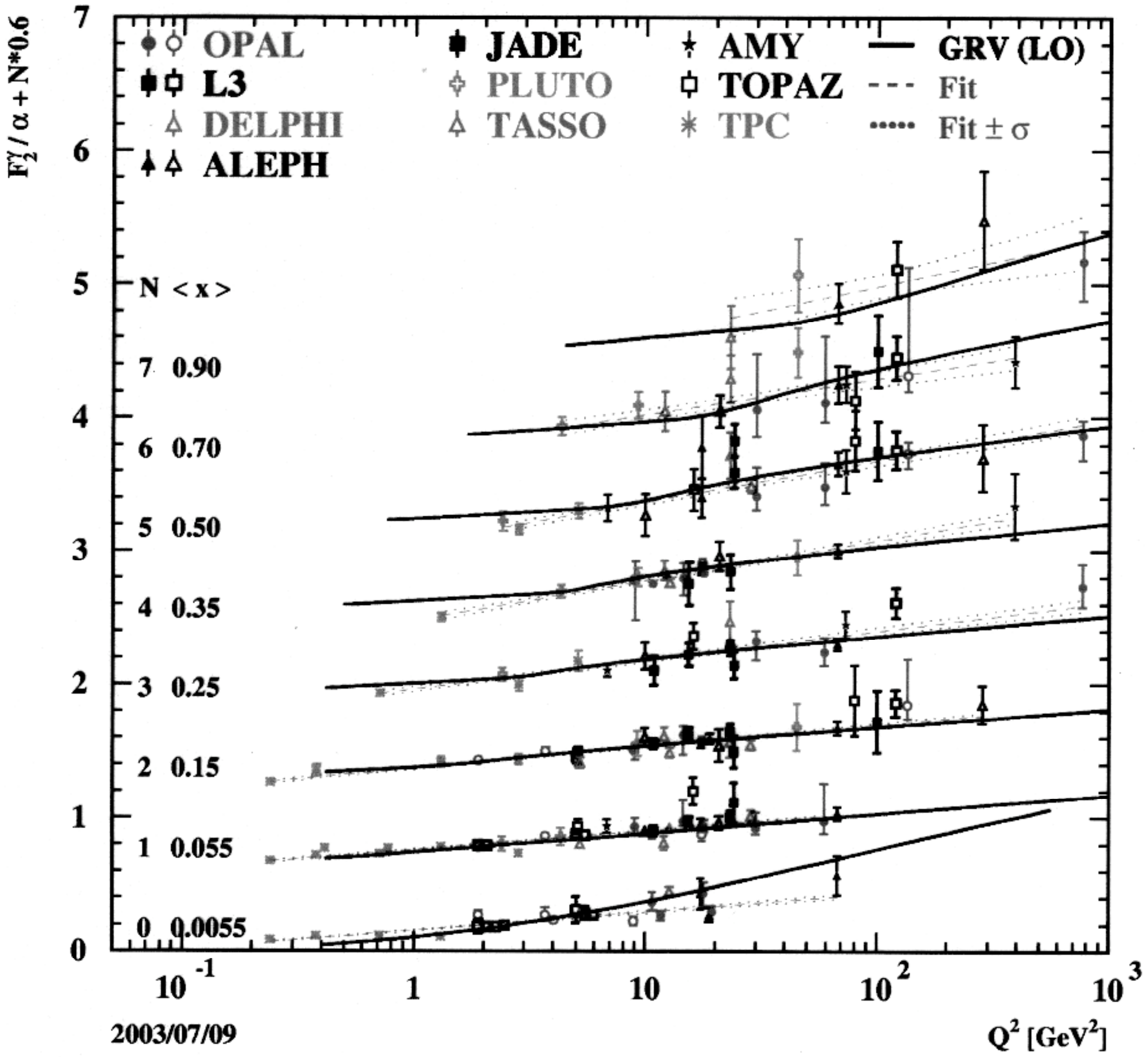


$Q^2 = 1000$ GeV^2

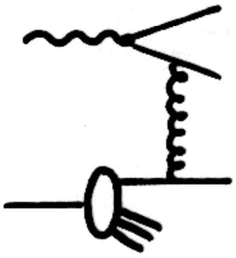



ZEUS
xf
ZEUS NLO QCD fit
$\alpha_s(M_Z^2) = 0.118$
tot. error
CTEQ 6M
MRST2001
Q^2=10 GeV^2
xu_v
xd_v
xg(× 0.05)
xS(× 0.05)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10^{-3}
10^{-2}
10^{-1}
1
x

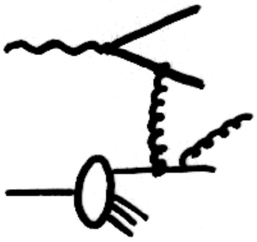

γ*
γ
=
γ*
q
γ


$F_2^\gamma / \alpha + N*0.6$
OPAL
L3
DELPHI
ALEPH
JADE
PLUTO
TASSO
AMY
TOPAZ
TPC
GRV (LO)
Fit
Fit ± σ
N <x>
7 0.90
6 0.70
5 0.50
4 0.35
3 0.25
2 0.15
1 0.055
0 0.0055
0
1
2
3
4
5
6
7
10^{-1}
1
10
10^2
10^3
2003/07/09
Q^2 [GeV^2]

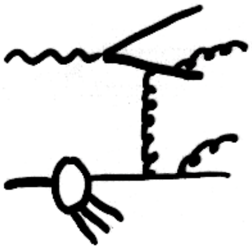
The small-x problem



$$\sigma \sim \alpha_s s^0$$



$$\sigma \sim \alpha_s^2 \log s$$



$$\sigma \sim \alpha_s^3 \log^2 s$$

$$e^y \sim \hat{s} \sim 2\nu \sim \frac{Q^2}{x}$$

large $\hat{s}$ <=> small x

<=> large y

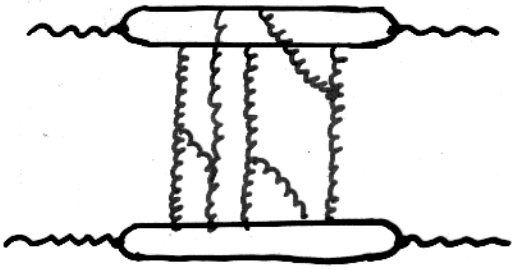
$$\sigma = \underbrace{\alpha_s \sum_n c_n (\alpha_s \log s)^n}_{\text{leading log}} + \alpha_s^2 \underbrace{\sum_n c'_n (\alpha_s \log s)^n}_{\text{subleading log}} + \dots$$

resum $\Rightarrow$ $\sigma \sim s^{\lambda}$ $xg(x) \sim x^{-\lambda}$

leading-log $\lambda = 12 \ln 2 \frac{\alpha_s}{\pi} \approx 0.5$

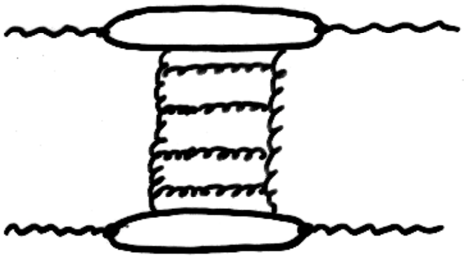
subleading logs $\lambda \simeq 0.3$

- huge correction
- i.R. sensitive



leading-log

BFKL



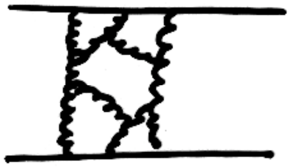
leading-log

DGLAP

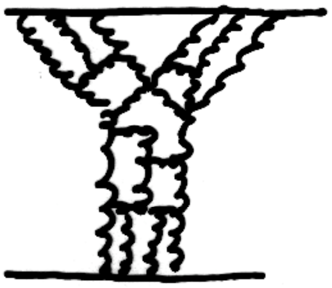
$\Rightarrow$ **Many gluons at small x!**

Saturation

many gluons => recombine



BFKL



recombination

$\sim x g(x)$

$\sim [x g(x)]^2$

=> set of equations describing
n ladders → m ladders

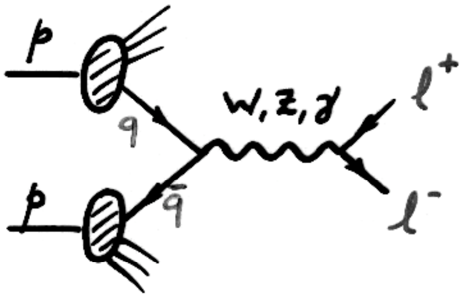
(Balitsky)

Simplest approximation

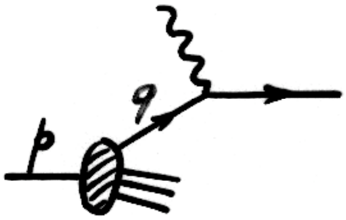
=> Balitsky - Kovchegov equations

- based on pert. theory

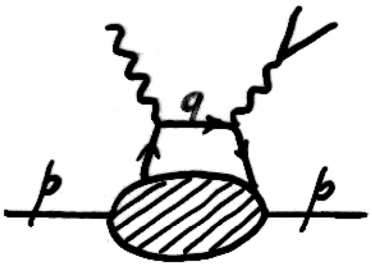
Associated processes:



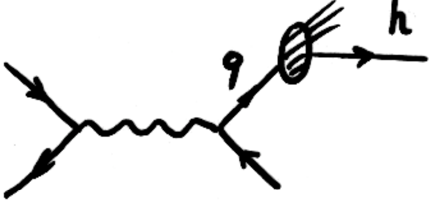
Drell - Yan



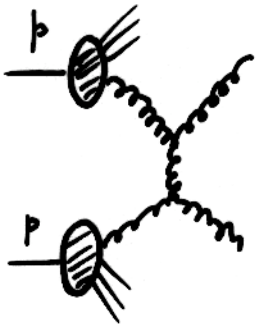
spin - dependent
structure functions



exclusive processes
OFPD

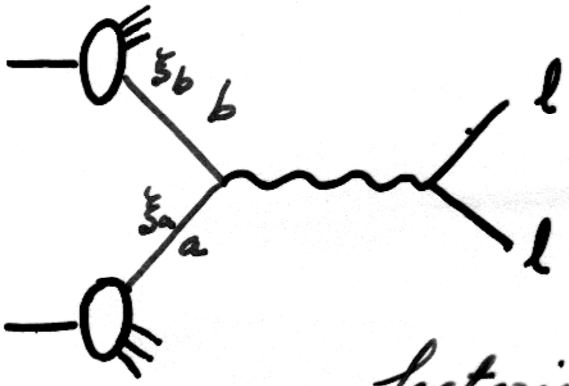


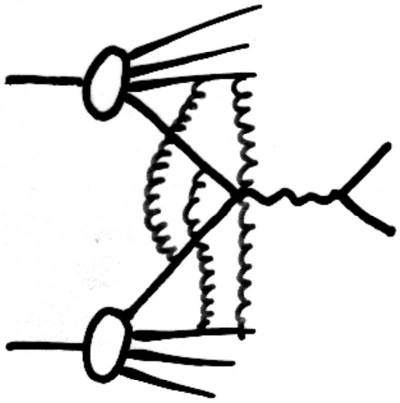
fragmentation functions



hadronic jets

Drell - Yan



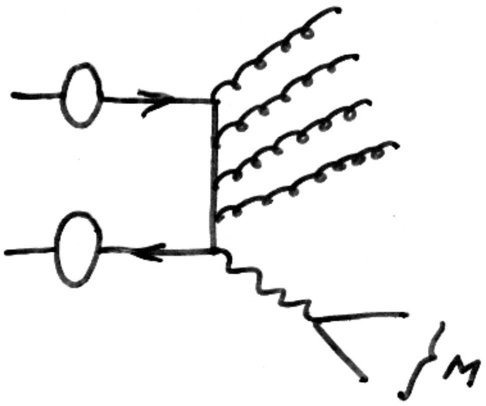


$f_a \quad f_b \xrightarrow[\text{theorem}]{\text{factorisation}}$ same as in DIS

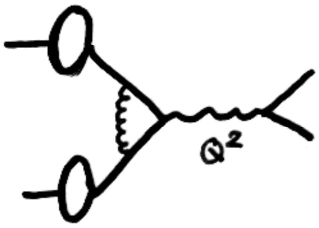
$$\frac{d\sigma}{dy} \simeq \sum \int_{x_a}^{1} d\xi_a \int_{x_b}^{1} d\xi_b \, f_a(\xi_a, \mu_F) f_b(\xi_b, \mu_F) \times \frac{d\hat{\sigma}_{ab}(\mu_F)}{dy}$$

p_T / E_T distribution

$\Rightarrow$ resum $\log\left(\frac{p_T}{M}\right)$



The K factor



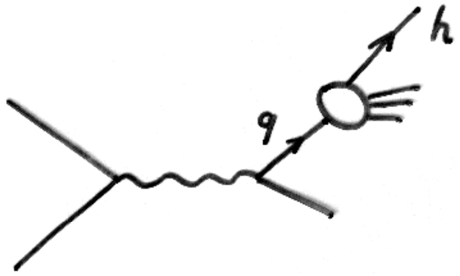
large correction:

$$K_1(Q^2) = 1 + \left(1 + \frac{4\pi^2}{3}\right) \frac{C_F \alpha_s(Q^2)}{2\pi}$$

$$K(Q^2) = K_\infty(Q^2) = e^{\pi C_F \alpha_s(Q^2)/2} \left[1 + \left(1 + \frac{\pi^2}{3}\right) \frac{C_F \alpha_s(Q^2)}{2\pi}\right]$$

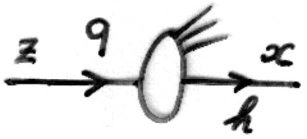
$K_1 \simeq 1.6$	$Q = 10\,\mathrm{GeV}$
$K_1 \simeq 1.5$	$Q = 40\,\mathrm{GeV}$
$K_1 \simeq 1.4$	$Q = 90\,\mathrm{GeV}$

Fragmentation functions



$$x = \frac{2E_h}{\sqrt{s}}$$

$$F_h \equiv \frac{1}{\sigma}\frac{d\sigma}{dx}(e^+e^- \to hx)$$



$$F_h = \sum_i \int_x^1 \frac{dz}{z} C_i^{(s)} D_i^{\,h}\left(\frac{x}{z}, s\right)$$

→ probability to find a hadron within a quark!

inverse process from DIS

- same evolution for lowest order with $P_{ab} \longrightarrow P_{ba}$
- different higher orders due to s-channel instead of t-channel

average multiplicity

$$\langle n_h \rangle = \int_0^1 dx\ F^h(x)$$