THE CASIMIR EFFECT

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Abstract

The Casimir effect is usually interpreted as due to the modification of the zero point energy of QED when two perfectly conducting plates are put very close to each other, and, consequently, as a proof of the "reality" of this zero point energy. The Dark Energy, necessary to explain the acceleration of the expansion of the Universe is sometimes viewed as another proof of the same reality. The usual interpretation of the Casimir effect is however challenged by some authors who rather consider it as a "giant" van der Waals effect. All these aspects are discussed.

1 Introduction

The Casimir effect corresponds to the force acting beetween two uncharged parallel condensor plates. It is customarily attributed to the change in zero point energy of the electromagnetic vacuum extending between the plates with respect to the one of the vacuum contained in the same region in the absence of plates. The zero point energy is supposed to result from the standard quantization of the free electromagnetic field. This energy is not directly observable, but the force between the two plates results from the change of the zero point energy contained between the plates when the latter are moved apart from each other. The Casimir effect is generally considered as a "proof" of the reality of the zero point energy. The dark energy, seemingly necessary to explain the observed accelerating expansion of the Universe is sometimes advocated as another "proof" of the same reality. All these aspects are shortly examined below.

2 The "usual" derivation of the Casimir effect

Let us consider an ideal condensor with infinite perfectly conducting plates in the x and y directions separated by a distance d along the z direction as shown in Fig. 1.

The interaction energy between the two plates can be defined as the difference between the zero point energies contained in the space between the planes, in the two respective configurations. This supposes that the field outside the cavity is not changed, which, of course, holds classically. For a discussion of this point, as well as for corrections due the finiteness of the plates, see Ref. [1]. In general, the zero point energy of the electromagnetic field is given

$$E_{cav} = \sum_k g_s(k) \frac{\hbar\omega_k}{2}, \tag{1}$$

where the sum runs over the normal modes of the field, where the ω_k's are the frequencies of these modes and where $g_s(k)$ is the degeneracy of the mode k , due to polarization.

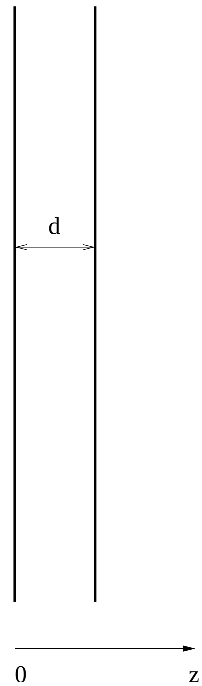

Fig. 1. *I*deal condensor with infinite extension in the x and y directions. The origin of the z axis is located on the left plate.

Let us start with the case of the cavity. In order to identify the modes more easily, let us consider, as asual, field configurations which are periodic in the x and y directions with "periods" L_x and L_y, respectively. The normal modes are determined by the boundary conditions on the surface of the conductors. We remind that the tangential electric field and the normal component of the magnetic field should vanish on these boundaries. These conditions are realized when the vector potential $\vec{\psi}$ is given, for $k_z \neq 0$, by

$$\vec{\psi} = \vec{\epsilon}\, e^{ik_x x} e^{ik_y y} sin k_z z\; e^{-i\omega_k t}, \tag{2}$$

with

$$k_x = n_x \frac{2\pi}{L_x}, \quad k_y = n_y \frac{2\pi}{L_y}, \tag{3}$$

where n_x and n_y are integer numbers and with

$$k_z = n_z \frac{\pi}{d}, \tag{4}$$

where n_z is a positive integer, the vector $\vec{\epsilon}$ is the polarisation vector and where $\vec{k} = (k_x, k_y, k_z)$. It is perpendicular to the vector $\vec{k}$: $\vec{\epsilon}.\vec{k} = 0$, as explained below. The quantity $\vec{\psi}$ can be viewed as the vector potential in the Coulomb gauge. The electric field is then proportional to the vector potential:

$$\vec{E} = i\omega_k \vec{\epsilon}\, e^{ik_x x} e^{ik_y y} sin k_z z\ e^{-i\omega_k t}. \tag{5}$$

where $\vec{\epsilon}$ is the polarization vector. The magnetic field can be written as

$$\vec{B} = \vec{k} \times \vec{\epsilon}\, e^{ik_x x} e^{ik_y y} cos k_z z\ e^{-i\omega_k t}. \tag{6}$$

The boundary conditions are satisfied as follows. The vanishing of the tangential electric field is guaranteed by the presence of the sine function in $\vec{\psi}$ and the values of k_z. The vanishing of the normal magnetic field requires

$$\vec{e_z}.(\vec{k} \times \vec{\epsilon}) = 0, \tag{7}$$

which should hold together with

$$\vec{k}.\vec{\epsilon} = 0, \tag{8}$$

the latter relation resulting from $\vec{\nabla}.\vec{E}$=0. The last two equations are explicitated as:

$$k_x \epsilon_y - k_y \epsilon_x = 0, \tag{9}$$

$$k_x \epsilon_x + k_y \epsilon_y + k_z \epsilon_z = 0. \tag{10}$$

For each value of $\vec{k}$ with $k_z \neq 0$, there is an infinity of solutions and it is thus possible to select two modes corresponding to two vectors $\vec{\epsilon}$ satisfying the last two equations and perpendicular to each other. The conditions are satisfied differently for the modes $\vec{k}$ with $k_z = 0$. Form (2) is not satisfactory, since ψ is then vanishing identically. However, the form

$$\vec{\psi} = \vec{\epsilon}\, e^{ik_x x} e^{ik_y y}\ e^{-i\omega_k t}, \tag{11}$$

where $sin(k_z z)$ has been replaced by $cos(k_z z)$ (equal to unity), is also a solution of the Laplace equation in this case. The electric and magnetic fields are given by

$$\vec{E} = i\omega_k \vec{\epsilon}\, e^{ik_x x} e^{ik_y y}\ e^{-i\omega_k t}. \tag{12}$$

and

$$\vec{B} = -\vec{k} \times \vec{\epsilon}\, e^{ik_x x} e^{ik_y y}\ e^{-i\omega_k t}. \tag{13}$$

The vanishing of the tangential electric field can now only be guaranteed by a normal $\vec{\epsilon}$ vector, which also guarantees the vanishing of the normal component of the magnetic field (13), since the vector $\vec{k}$ has only tangential component. However, the vector $\vec{\epsilon}$ does still have to fulfill Eqs. (9,10). For $k_z = 0$, the only solution is $\vec{\epsilon} = \vec{e}_z$: for these modes, there is only one possible polarisation (g_s=1). These $k_z = 0$ modes are often forgotten in the literature (see for instance Ref. [2]) , leading to confusing statements about regularisation procedures.

The energy of the cavity for the $k_z \neq 0$ modes (indicated by the prime) is given by

$$E'_{cav} = \hbar c \sum_{n_x=-\infty}^{+\infty} \sum_{n_y=-\infty}^{+\infty} \sum_{n_z=1}^{\infty} \left[\left(\frac{n_x 2\pi}{L_x}\right)^2 + \left(\frac{n_y 2\pi}{L_x}\right)^2 + \left(\frac{n_z \pi}{d}\right)^2 \right]^{1/2}. \tag{14}$$

As usual, one replaces the summation on n_x and n_y by integration on continuous variables:

$$E'_{cav} = \hbar c L_x L_y \int_{-\infty}^{+\infty} dn_x \int_{-\infty}^{+\infty} dn_y \sum_{n_z=1}^{\infty} \left[\left(\frac{n_x 2\pi}{L_x}\right)^2 + \left(\frac{n_y 2\pi}{L_x}\right)^2 + \left(\frac{n_z \pi}{d}\right)^2 \right]^{1/2}, \tag{15}$$

which causes no difference when the limit $L_x, L_y \to \infty$ is taken. Changing variables from n_x, n_y to k_x, k_y, dividing the expression by $L_x L_y$ and taking the limit of L_x and L_y tending to infinity, yields for the energy per unit area:

$$\frac{E'_{cav}}{S} = \hbar c \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \left[k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2 \right]^{1/2}. \tag{16}$$

Finally, one integrates over the angle of the wave vector in the $x - y$ plane, introduces the auxiliary variable $u = ((k_x^2 + k_y^2)d^2)/\pi^2 = k_\perp^2 d^2/\pi^2$ and obtains

$$\frac{E'_{cav}}{S} = \frac{\hbar c \pi^2}{4d^3} \sum_{n=1}^{\infty} \int_0^{\infty} du \left(u + n^2\right)^{1/2}. \tag{17}$$

We still have to add the contribution of the $k_z = 0$ (or $n = 0$) modes. One has:

$$\frac{E_{cav}}{S} = \frac{\hbar c \pi^2}{4d^3} \left[\sum_{n=1}^{\infty} \int_0^{\infty} du \left(u + n^2\right)^{1/2} + \frac{1}{2} \int_0^{\infty} du \sqrt{u} \right]. \tag{18}$$

This expression is divergent, as is the similar expression for the energy of the free field. It may be hoped that the difference between the two expressions is finite.

We need the value of the energy of the free field in the volume of the cavity. In general, the energy per unit volume is given by [1]

$$\frac{E_{free}}{V} = \hbar c \int \frac{d^3\vec{k}}{(2\pi)^3} k. \tag{19}$$

It is advantageous to rewrite this expression as

$$\frac{E_{free}}{V} = \frac{\hbar c}{(2\pi)^3} \int d^2\vec{k}_\perp \int dk_z \sqrt{k_\perp^2 + k_z^2}, \tag{20}$$

or as

$$\frac{E_{free}}{V} = \frac{\hbar c}{(2\pi)^2} \int_0^\infty k_\perp dk_\perp \int_{-\infty}^{+\infty} dk_z \sqrt{k_\perp^2 + k_z^2}. \tag{21}$$

Taking account of the fact that the integrand is an even function of k_z and introducing the auxiliary variables $u = k_\perp^2 d^2/\pi^2$ and $x = k_z/d$ lead to

$$\frac{E_{free}}{V} = \frac{\hbar c \pi^2}{4d^4} \int_0^\infty dx \int_0^\infty du \sqrt{u + x^2}. \tag{22}$$

The energy of the free field in the volume of the cavity is thus obtained by multiplying this expression by the volume V (equal to Sd). One then obtains for the change in the zero point energy per unit surface:

$$\frac{\Delta E}{S} = \frac{E_{cav}}{S} - \frac{E_{free}}{S} = \frac{\hbar c \pi^2}{4d^3} \{ \frac{1}{2} \int_0^\infty du \sqrt{u} + \sum_{n=1}^\infty \int_0^\infty du \left(u + n^2 \right)^{1/2} - \int_0^\infty dx \int_0^\infty du \left(u + x^2 \right)^{1/2} \}. \tag{23}$$

All terms in the rhs are divergent. We will come to this problem soon. It is remarkable that this expression involves the difference between the integral from zero to infinity of the function $f(x) = \int_0^\infty du \left(u + x^2\right)^{1/2}$ and the sum of the values of this function on the positive integers. There is a famous theorem by Euler and McLaurin connecting these quantities, in general. It states that:

$$\sum_{n=1}^\infty f(n) = \int_0^\infty f(x) dx - \frac{1}{2} [f(0) + f(\infty)] + \frac{1}{12} [f'(\infty) - f'(0)] - \frac{1}{720} [f'''(\infty) - f'''(0)] + \cdots \tag{24}$$

where the dots indicates similar terms for higher order odd derivatives. The coefficients in front of the brackets are related to the Bernoulli numbers B_i: they are equal to $-B_{2k}/(2k)!$ for the term involving the derivatives of order $2k - 1$. See Ref. [3].

[1] The $\vec{k} = 0$ mode does not pose any worry, since its contribution is vanishing.

We can write the function $f(x)$ mentioned above as

$$f(x) = \int_0^\infty du \left(u + x^2\right)^{1/2} = \int_{x^2}^\infty dt\sqrt{t}. \tag{25}$$

Formally, considering the dependence upon x through the lower bound of the integral only, one has:

$$f'(x) = -2x^2, f''(x) = -4x, f'''(x) = -4 \tag{26}$$

and all higher derivatives are vanishing. So, retaining the single nondivergent term (which corresponds to $f'''(0)$), one finally obtains:

$$\boxed{\frac{\Delta E}{S} = -\frac{\hbar c \pi^2}{720 d^3}.} \tag{27}$$

This is the expression of the Casimir effect, which looks universal and which depends only upon the two fundamental constants $\hbar$ and c and on the distance d.

Let us comment on the divergence problems first. Of course, expression (23) and the Euler-MacLaurin theorem apply when the quantities are convergent. However, one can make the final result meaningful by regularizing the integral and the sum. The regularisation at infinity is not a problem. It is easy to introduce a suitable integration factor. For instance, it is easy to see that all terms at infinity vanish owing to the substitution

$$f(x) \to \int_0^\infty du \left(u + x^2\right)^{1/2} e^{-(u+x^2)\alpha}, \tag{28}$$

The terms corresponding to higher order derivatives (at $x = 0$ as well as at $x = \infty$) are finite and vanish as $\alpha \to 0$. Similarly, the first and third derivatives at $x = 0$ are incremented by quantities that vanishes as $\alpha \to 0$. Actually, the regularisation can be achieved by any cut-off function which decreases sufficiently rapidly at large x (non necessarily exponentially), which goes to unity as x goes to zero with vanishing derivatives, like the functions $g(x) = 1 - exp(-a/x)$.

3 Which force?

The force (per unit surface) acting between the plates is given by

$$\frac{F}{S} = -\frac{\hbar c \pi^2}{240 d^4}. \tag{29}$$

The negative sign corresponds to an attractive force. It is a tiny force. For instance, at d=1 μm, it amounts to F/S= $4\times10^{-4}N/m^2$. Of course, due to the fourth power, it increases very rapidly as the distance decreases. At d=1 nm, the force reaches F/S= $4\times10^{8}N/m^2$.

Needless to say that the experimental verification of the Casimir effect has taken quite a long time. Among the unsuccessful trials, one should mention the experiment by Sparnaay [4], which although unsuccessful, has nevertheless identified the main difficulties: a perfect parallelism of the plates, a lack of impurities (which may scatter the normal modes) and the elimination of the residual charges. Let us also mention the experiment of Derjaguin et al [5], who were the first to obtain a meaningful result, verifying the predictions at the 60% level, before the experiments by Lamoreaux [6] and Ederth [7] who verified the theoretical value with an accuracy of ∼1%.

4 The Casimir force and the van der Waals effect

4.1 Introduction

The Casimir force may be viewed as a quantum interaction between two neutral objects. Of course, the conducting properties of these objects should be taken into account at some point. But for the moment, let us consider the Casimir force as the force acting between macroscopic neutral objects and address the question whether there is some relationship with the force acting in another system of this kind, namely the system of two neutral atoms. We will examine this question in a bit historical perspective, which helps to understand the relationship.

4.2 The van der Waals force in the simplest approach

The van der Waals interaction has been calculated microscopically for the first time by London [8]. The hamiltonian of the system can be written as

$$\begin{aligned} &H = H_1 + H_2 + H', \\ &H' = \frac{Z_1 Z_2 e^2}{|\vec{R_1} - \vec{R_2}|} - \sum_{i=1}^{Z_1} \frac{Z_2 e^2}{|\vec{R_1} + \vec{r_i} - \vec{R_2}|} - \sum_{j=1}^{Z_2} \frac{Z_1 e^2}{|\vec{R_2} + \vec{r_j} - \vec{R_1}|} + \sum_{i=1}^{Z_1}\sum_{j=1}^{Z_2} \frac{e^2}{|\vec{R_2} + \vec{r_j} - \vec{R_1} - \vec{r_i}|}. \end{aligned} \tag{30}$$

In this equation, Z_1 and Z_2 are the charge numbers of the nuclei and $\vec{r_i}, \vec{r_j}$ are the coordinates of the electrons with respect to the position of the respective nuclei. Expanding H' up to second order in the electron coordinates $\vec{r_i}, \vec{r_j}$ (which is presumably sufficient if the distance $r = |\vec{R_1} - \vec{R_2}|$ is large in comparison with the atomic sizes), one has

$$H' = -2\frac{Z_2 e}{|\vec{R_1} - \vec{R_2}|^3}(\vec{R_1} - \vec{R_2}).\sum_{i=1}^{Z_1} e\vec{r_i} - 2\frac{Z_1 e}{|\vec{R_1} - \vec{R_2}|^3}(\vec{R_2} - \vec{R_1}).\sum_{j=1}^{Z_2} e\vec{r_j}. \tag{31}$$

Considering H' as a perturbation, the change in energy of the ground state can be calculated by standard perturbation theory. The first order contribution vanishes. The second order contribution writes:

$$\Delta E^{(2)} = -\frac{6e^4}{r^6} \sum_{k \neq 0} \sum_{l \neq 0} \frac{|\langle k \mid \sum_i \vec{r}_i.\vec{n} \mid 0 \rangle|^2 |\langle l \mid \sum_j \vec{r}_j.\vec{n} \mid 0 \rangle|^2}{E_{1k} - E_{10} + E_{2k} - E_{20}}. \tag{32}$$

In this equation, k (l) labels the excited states of the first (second) atom, $|0\rangle$ is the ground state of the atoms (we avoided to put an indice recalling which atom is concerned, since there is no risk of confusion) and $\vec{n}$ is the unit vector along the line joining the two nuclei (the direction is irrelevant). The sums run, in principle, over all the excited states, but in practice only on those which are connected to the ground state, by off-diagonal matrix elements of the dipole moments $\vec{d}_1 = \sum \vec{r}_i$ or $\vec{d}_2 = \sum \vec{r}_j$. For atoms with $J = 0$ ground states, i.e. with spherical shapes, the sum in Eq. 32 is limited to $J = 1$ states, but involves a summation over the magnetic quantum numbers. Using the Wigner-Eckart theorem to relate the matrix elements of different magnetic quantum numbers, it is easy to see that that the quantity $\Delta E^{(2)}$does not depend upon the orientation of the vector $\vec{n}$, as intuitively expected.

The physical meaning of Eq. (32) is rather clear. Classically, two neutral objects with spherical symmetry, even if they are locally charged have no Coulomb interactions. All their multipole moments are vanishing. Quantum mechanically, spherical neutral atoms, have zero electric dipole moments only on the average. They are fluctuating. There is a non vanishing probability for having the two atoms with non zero dipole moments and therefore experiencing a Coulomb interaction. The van der Waals force is thus a purely quantum force originating from quantum fluctuations.

It may be of interest to make two remarks. First, Eq. (32) is not valid if r is of the order of the size of the atoms. At short distances, the interaction should be repulsive, due to the Pauli principle: the latter forbids to put simply electrons at the top of each other; this is only possible if the electrons of one atom are put at unoccupied orbits of the other, which requires a strong increase of the kinetic energy. The repulsive nature of the atom-atom interaction at short distance is often embodied by Lennard-Jones potentials. Second, it may be worthwhile to notice that the expression in Eq. (32) is almost but not exactly proportional to the product of the electric polarisabilities of the atoms. The electric polarisability is defined as the the ratio between the induced electric dipole acquired by an atom in a static electric field (considered as uniform for simplicity) and the magnitude of this electric field. It is given in second order by

$$\alpha = \sum_{k \neq 0} \frac{|\langle k \mid \sum_i e\vec{r}_i.\vec{n} \mid 0 \rangle|^2}{E_k - E_0}. \tag{33}$$

For a matter made of these atoms, the dielectric constant ε is given, in the dilute limit, as

$$\varepsilon = 1 + 4\pi\alpha n_{at}, \tag{34}$$

where n_{at} is the density of atoms.

When he was working at the Philips company in Eindhoven, Casimir got interested into the behaviour of the van der Waals interaction at large distances. Two colleagues of him, Verwey and Overbeek, were studying experimentally colloidal suspensions. It seems that a simple model based on the van der Waals force successfully reproduced their observations, but failed for dilute suspensions. They interpreted their observation as due to a weakening of the van der Waals force at large distance [9–11]. They were thinking that retardation effects were the cause of this weakening. If the van der Waals interaction is interpreted as due to the interaction between fluctuating dipoles, the fluctuation of one dipole takes some time before influencing the other atom when the interdistance is large enough and vice-versa. They approached Casimir and asked him whether he could calculate this effect. A little bit later, the answer was given in paper by Casimir and Polder [12]. We are not going to enter in the details of this complicated calculation, but we will give the general ideas and the results.

In Eq. (32), only the static (Coulomb) interactions are introduced. How to cope with retardation effects in Quantum Mechanics? Retardation effects are linked with the perturbations of the radiation field which propagate at finite speed. One has thus to introduce this radiation field. This is usually done by introducing a time-dependent vector potential in the hamiltonian. Adopting the Coulomb gauge and the minimum substitution principle, this is equivalent to replace the momentum of the electron $\vec{p_i}$ by $\vec{p_i} + \frac{e}{c}\vec{A}(\vec{r_i}, t)$. With such a prescription, one has to add, along with H', a second perturbation of the form:

$$H'' = \frac{e}{mc}\sum_i \vec{A}(\vec{r_i}, t).\vec{p_i} + \frac{e^2}{2mc^2}\sum_i \vec{A}^2(\vec{r_i}, t). \tag{35}$$

The procedure is fairly standard. The system of the two atoms with a relative distance d is enclosed in a cubic box with perfectly conducting walls. The vector potential is written as an expansion on the normal modes

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}}\sum_s (a_{\vec{k},s}e^{-i\omega t} + a^\dagger_{\vec{k},s}e^{i\omega t})\vec{X}_{\vec{k},s}(\vec{r}), \tag{36}$$

where $\vec{X}_{\vec{k},s}$ is the vector field characteristic of the mode $\vec{k}, s$, duly normalized. The operators a and $a^\dagger$ are the usual destruction and creation operators. The total hamiltonian is written as:

$$H = H_1 + H_2 + H_{rad} + H' + H'' = H_0 + H' + H''. \tag{37}$$

where H_{rad} is the free radiation hamiltonian. The change in the ground state energy of the total system, due to the perturbation H' and H'', is then calculated to second order. Note that to be consistent, the calculation should include second order terms in the first part of H'' (linear in a and $a^\dagger$) and first order terms in the second part of this operator as it is already quadratic in a and $a^\dagger$. Finally, the size of the box is extended to infinity, while keeping the distance d fixed. Needless to say that the calculation is rather cumbersome. For detail, we refer to the original paper. We simply quote the results.

For small r (actually smaller than the absolute value of the non-diagonal matrix elements of the dipole operator), the London result (Eq. 32) is recovered. For large r (in principle larger than the above-mentioned quantities), the following simple results is obtained:

$$\Delta E^{(2)} = -\frac{23\hbar c}{4\pi r^7}\alpha_1\alpha_2. \tag{38}$$

The van der Waals interaction is weakening as the distance increases and factorises in the polarisabilites of the atoms.

In the same paper, Casimir and Polder investigated also the interaction of an atom with a conducting wall. The principle is the same: put the atom in the box at a fixed distance d from a wall and let the size of the box become infinite while keeping a wall fixed. In this case, the hamiltonian H' is the Coulomb interaction between the atom and the wall, which is taken as the interaction of the dipole moment of the atom and its image. The dipole moment is then considered as the corresponding quantum operator. Note that the Coulomb atom-wall is neglected in the case of two atoms, since the walls are eventually removed to infinity. It is interesting to quote the results. For small distances, the change of energy is given by:

$$\Delta E^{(2)}_{atom-wall} = -\frac{3}{8d^3}\sum_k |\langle k \mid \sum_i e\vec{r}_i.\vec{n} \mid 0\rangle|^2, \tag{39}$$

where $\vec{n}$ is the unit vector perpendicular to the plane. For large distances, one has

$$\Delta E^{(2)}_{atom-wall} = -\frac{\hbar c}{8\pi}\frac{3\alpha_1}{d^4}, \tag{40}$$

where α_1 is the polarisability of the atom.

Casimir was intrigued by the simplicity of the results, especially the one of Eq. 38 and was wandering whether they can be more general. After all, these results were derived using the standard apparatus of perturbation theory (to second order in the fine-structure constant α, see later). He once discussed these results with Niels Bohr, who is said to have replied [13]: “Why don’t you calculate the effect by evaluating the difference of zero point energies in the electromagnetic field?”. Of course this requires to calculate the normal modes in the presence of the atoms, which is very hard. Casimir realized that the calculation could be more easily performed for the case of the cavity as it is done in Section 2 and published his result in Ref. [14].

5 The nature of the Casimir force

5.1 Introduction

Although the existence and magnitude of the Casimir effect is now well established, there is still some controversy concerning its nature and its interpretation. The Casimir effect looks universal. Formula (27) indeed solely depends upon the the constants $\hbar$ and c and upon the interdistance d. It is therefore considered as a property of the electromagnetic vacuum, modified by the presence of the condensor plates. On the other hand, it is tempting to interpret the Casimir effect as a generalized van der Waals interaction between two gigantic "molecules", the conducting planes. In this perspective, the Casimir effect is reduced to an ordinary (though quantal) electromagnetic effect. It is then surprising that this effect is not dependent upon the fine structure constant α. Actually, it can be shown that the independence upon α results from the implicit hypothesis of perfectly conducting planes. When this hypothesis is released, correcting terms in α should be added. The result (27) appears to be correct in the the limit of very large values of α. In the following we will give simple arguments supporting this assertion. We will also discuss the relation between the Casimir effect and the quantum fluctuations of the electromagnetic vacuum. Since this question is still under debate, we will limit ourselves to general considerations.

5.2 The dependence of the Casimir effect on the fine structure constant

Actual metals are not perfectly conducting. They are characterized basically by two quantities: the plasma frequency ω_{pl} and the skin depth δ. For frequencies above ω_{pl}, the conductivity basically goes to zero. The quantity δ measures the distance up to which electromagnetic waves penetrate the metal. A perfect conductor is characterized by infinite ω_{pl} and $\delta = 0$. In actual metals, ω_{pl} and $1/\delta$ depend upon the fine structure constant α and vanish when $\alpha \to 0$. We turn to the simplest model for real metals, namely the Drude model, to describe qualitatively what is happening. We closely follow here Ref. [15]. Basically, in the Drude model, electrons are moving independently under the influence of the electric field and they are subject to a friction force. Let $E = E_0 e^{-i\omega t}$ be the applied electric field. The Newton equation of motion for the electron can be written as:

$$m_e \frac{d^2 x}{dt^2} = -eE_0 e^{-i\omega t} - \gamma \frac{dx}{dt}. \tag{41}$$

where m_e is the electron mass and γ is the friction parameter. The solution is an oscillatory function of time $x(t) = x_0 e^{-i\omega t}$, with

$$x_0 = \frac{eE_0}{m_e \omega(\omega + i\overline{\gamma})}, \tag{42}$$

where we have introduce the reduced friction parameter $\overline{\gamma} = \gamma/m_e$. It is then easy to calculate the induced current ($j = -endx/dt$). The result gives readily the conductivity ($\sigma = j/E$) under the form

$$\sigma = \frac{e^2 n}{m_e} \frac{1}{\overline{\gamma} - i\omega}, \tag{43}$$

where n is the electron density. The plasma frequency is given by

$$\omega_{pl}^2 = \frac{4\pi e^2 n}{m_e}. \tag{44}$$

The skin depth, which is defined by

$$\delta^{-2} = \frac{2\pi\omega|\sigma|}{c^2}. \tag{45}$$

in general, becomes in the Drude model

$$\delta = \frac{c}{\left(\frac{1}{2}\frac{\omega\omega_{pl}^2}{\sqrt{\overline{\gamma}^2+\omega^2}}\right)^{\frac{1}{2}}}. \tag{46}$$

In this model, indefinitely increasing ω_{pl} automatically implies $\delta \to 0$. In practice, typical frequencies of interest are larger than $\overline{\gamma}$, and δ becomes

$$\delta \approx \frac{c}{\omega_{pl}\sqrt{2}}. \tag{47}$$

The frequencies that are relevant for the Casimir effect are those with a frequency smaller than c/d. The perfect conductor approximation requires therefore that $c/d \ll \omega_{pl}$. Combining this relation with Eq. 44 gives the following condition:

$$\alpha \gg \frac{m_e c}{4\pi\hbar n d^2}. \tag{48}$$

For typical cases (copper plates separated by a micrometer), the rhs is of the order of 10^{-5}. Condition (48) is comfortably satisfied by the physical value of α. The standard Casimir result can then be regarded as the $\alpha \to \infty$ limit of the true result which is dependent on the nature of the metal. For large α, one expects corrections to the Casimir result which could be put in series of negative powers of α. This result may be obtained very roughly by saying that for real metals, the limits of the cavity become somewhat transparent to the electromagnetic field and that the effective width of the cavity becomes $d + 2\delta$. One thus expects, instead of the relation (27)

$$\frac{\Delta E}{S} \approx -\frac{\hbar c\pi^2}{720(d+2\delta)^3} \approx \frac{\hbar c\pi^2}{720d^3}\left(1 - \frac{6\delta}{d} + ...\right). \tag{49}$$

In this equation, the dots indicate higher order powers in $2\delta/d$. Owing to Eqs. 47,44, it is easily seen that the latter ratio is proportional to $1/\sqrt{\alpha}$. The corrections thus disappear as $\alpha \to \infty$. It is also interesting to verify that the second term in the parenthesis of Eq. 49 becomes negligible compared to unity when condition (48) is fulfilled.

It is also interesting to look at the $\alpha \to 0$ limit. This limit is a little bit tricky as the typical size of atoms, the Bohr radius $\hbar^2/m_e e^2$, scales as $1/\alpha$. Therefore, n scales as α^3, ω_{pl} scales as α^2 and δ goes as $1/\alpha^2$. At very low α, the plates become transparent to the radiation and the Casimir effect goes away as $\alpha \to 0$. At low α, the Casimir effect is expected to be put in a series of increasing positive powers of α.

As all ordinary electromagnetic effects, the Casimir effect goes away when the fine structure constant goes to zero. The distinctive feature of the Casimir effect is that it reaches a finite value as $\alpha \to \infty$.

5.3 The Casimir effect: vacuum property or interaction between neutral objects?

The Casimir effect is often pointed as an evidence of the reality of quantum fluctuations of fields in vacuum. Just to quote a typical example, Weinberg in his introduction of the cosmological constant problem states [16]:

> Perhaps surprinsingly, it was a long time before particle physicists began seriously to worry about [quantum zero point fluctuation contribution to Λ] despite the demonstration in the Casimir effect of the reality of zero-point energies.

There are more recent quotations of this type. In his book "Particle Astrophysics", Perkins says [17]:

> That this concept [the vacuum energy] is not a figment of the physicist's imagination was already demonstrated many years ago, when Casimir predicted that by modifying the boundary conditions on the vacuum state, the change in the vacuum energy would lead to a measurable force, subsequently detected and measured by...

This kind of statements should be appraised after a close examination of the meaning of the expressions "reality" and "quantum fluctuations". Let us start with the second one. The vacuum can be considered as the quantum ground state of a field, say the electromagnetic field in the case of our discussion. There is little doubt that there are quantum fluctuations of observables associated to this field in the ground state, as it is for any observable which does not commute with the hamiltonian. The simplest observables are the electric and magnetic fields themselves. The expression "quantum fluctuations" is also used to denote the zero point energy of the vacuum which is interpreted as the energy "generated" by the quantum fluctuations. Had the electromagnetic field been vanishing with certainty, would it be natural to expect a vanishing energy. When one relates the Casimir effect to the "quantum fluctuations", it is to the second meaning of these words that one refers.

When boundaries are imposed to the electromagnetic field, the latter is changed. The question arises whether the ground state (the vacuum) of the electromagnetic field is changed. Physicists have been reluctant for a long time to admit that the energy of the vacuum is changed (advocating that the zero point energy is infinite and thus that its physical meaning is suspicious). The experimental measurements of the Casimir effect have given support to the idea that the zero point energy is perhaps unphysical, because it cannot be measured directly, but its variations when the geometry is changed are physical since they are observed. Before discussing this point, let us mention that nobody questions the change in the fluctating properties of the electromagnetic field when boundaries are introduced. We will come to this question later.

Let us examine the reality of the change in the zero point energy as revealed by the Casimir effect. The experiments are realized with condenser plates which, even in the limit of perfect conductors, are not merely a "mathematical" device serving to confine the electromagnetic field in a restricted region of space. They are composed of atoms or molecules which interact with the electromagnetic field. The force which is measured is in fact the force between the material plates. In some sense, it can be viewed as a van der Waals interaction between two gigantic molecules. It is the point of view adopted by many physicists, who consider that the usual calculation, based on the change in $\sum \hbar\omega$, is heuristic [15]. In other words, it is an accident that it gives the expression of the force between two conductors. A similar example is provided by the energy of a smooth charge in classical electrostatics which is given by $W = 1/2 \int d^3\vec{r} \int d^3\vec{r}' \rho(\vec{r})\rho(\vec{r}')/|\vec{r} - \vec{r}'|$ or, also, by the energy of the electric field $W = 1/(8\pi) \int d^3\vec{r} |\vec{E}(\vec{r})|^2$. This second expression cannot be viewed as an evidence of the "reality" of the electric field but as an alternative expression of the self-interaction of the charges which, heuristically, gives the correct magnitude of this self-interaction energy. The reality of the field and its extension outside of the sources cannot be proven by the action on a test charge as it is often stated in elementary courses. By looking at the effect of a test charge, one probes the interaction between the original source and the test charge. Bringing a test charge "changes the nature of the problem". The reality of the electromagnetic field exists, but is revealed by other kinds of phenomena like the Hertz experiment or the pair creation in a Coulomb field.

Let us now examine the relation between the van der Waals effect and the Casimir effect at the light of fluctuations of the vacuum, in the sense of fluctuations of operators in the vacuum. The van der Waals interaction between two atoms is the change of the energy in the system of the two atoms due to presence of the Coulomb interaction, which couples to the fluctuating dipoles of the atoms (this is translated as the interaction to the second-order perturbation term). The Casimir-Polder interaction is the change of the energy of the system made of the two atoms and the ground state of the electromagnetic field, due to the presence of the Coulomb interaction, which couples to the fluctuating dipoles of the atoms as before, but also due to the interaction between the electrons and the electromagnetic field, which couples to the fluctuations of the latter in the vacuum. When one of the atoms is replaced by a conducting plate, the fluctuations of this "atom" somehow disappear, but the interaction is given by the change in energy of the whole system (atom, wall and electromagnetic field), due to the same causes. The change in dimension of the system is reflected through the change in the power in the dependence of the effect upon the distance. When the second atom is replaced by the second plate, the interaction is solely due to the interaction of the plates with the fluctuating electromagnetic field.

It is interesting to note that the interaction between two conducting plates can be constructed without reference to zero point energy. First, we want to mention the method invented by Lifshitz [18,19] to calculate the interaction between dielectrics. The starting point is the quantum fluctuations of the electromagnetic field in large bodies. It is first argued that the fluctuations of the field are linked to the fluctuations of the polarisation density, that the fluctuations at different points (understood as involving scale larger than the atomic size) are uncorrelated and that the mean square of the fields at any given point is fixed by the change of energy implied by the appearance of a dielectric constant. The Maxwell stress tensor can be calculated and forces may be derived by taking derivatives. The formalism is cumbersome and has not been published totally (which has hampered interest in this kind of approach). Let us just quote some results. Lifshitz obtained an explicit expression for the force between two infinite dielectric bodies of dielectric constant ε with plane surfaces facing each other at a distance d (expression (90.1), p. 369 of Ref. [20]). From this expression, several limits can be obtained. The limit $\varepsilon \to 1$ corresponds to the dilute limit and the interaction between two molecules can be obtained. The perfect conductor limit is obtained as $\varepsilon \to \infty$ (this corresponds to the vanishing of the electric field inside the bodies). All the results of Section 3 and the Casimir formula can be obtained this way.

Let us also mention that the Quantum Field Theory can be formulated without any reference to zero point energy. For instance, the Casimir effect may be calculated (perturbatively) in terms of Feynmann diagrams with external legs, i.e. in terms of S-matrix elements making no reference to the vacuum. We refer to Ref. [15] for more information.

In conclusion there is a strong division between physicists regarding the interpretation of the Casimir effect. For many of them, the latter is a manisfestation of the vacuum energy. For many others, the Casimir effect is the interaction between two large polarisable bodies. It does not tell upon the quantum fluctuations more than any one-loop effect in quantum electrodynamics, like the vacuum polarisation of the Lamb shift.

6 The Casimir effect in Cosmology

It has been suggested that the Casimir effect or rather the vacuum energy could account for dark energy. The first suggestion dates to Einstein who introduced a cosmological constant in his fundamental equations of general relativity coupling the structure of space-time and the energy content:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + E_v g_{\mu\nu}). \tag{50}$$

Einstein introduced the last term in the rhs, or equivalently $\Lambda g_{\mu\nu}$, with the cosmological constant $\Lambda = 8\pi G E_v$, in order to manage a static solution. After the discovery of the expansion of the Universe, he dropped this term, that he considered as "the greatest mistake in [his] life". Later, with the development of quantum field theory, the possibility of the existence of a vacuum energy was taken seriously (it seems that Zeldovich was the first person to formulate such a possibility [21]) , especially after the experimental verification of the Casimir effect. It was even taken more seriously when it was realized that the expansion of the Universe seems to be accelerating [22,23]. Recent observations require a value of $\Lambda = (2.14 \pm 0.13 \times 10^{-3} eV)^4$ at the present time [24]. The name "dark energy" has been dubbed for this energy. The zero point energy of the electromagnetic field has been proposed as a candidate for the dark energy. We have given serious reservations on the reality of the zero point energy, but this suggestion meets difficulty beyond these reservations. First of all, the zero point energy density is in principle infinite. Of course, one may admit that the contribution of the very high fequencies should be cut somewhere, rendering the energy density finite. The natural cut should come from gravity and can be taken as the Planck scale. One then has

$$\rho_v = \hbar c \int^{k<k_{cut}} \frac{d^3\vec{k}}{(2\pi)^3} k = \hbar c \pi k_{cut}^4. \tag{51}$$

If k_{cut} is taken as the inverse of Planck length $\lambda_{PL} = 1.2 \times 10^{19}$ GeV, one obtains a vacuum energy of the order of 10^{121} GeV fm^{-3}. This should be compared to the critical energy density $\rho_c \approx 5$ GeV fm^{-3} and the part of about 75% taken by the dark energy. The supposed electromagnetic vacuum energy is thus enormously too large and there is no clear mechanism to reduce it. Furthermore, one should add in principle the contribution of the vacuum energy for the other fundamental fields. This has led to the crisis of the cosmological constant and to a serious questioning about the reality of the vacuum energy of the fields. Furthermore, there are plenty of condensates in the standard model which also contribute to dark energy in principle. The conclusion is that the (perturbative) vacuum energy (of the fields) has probably no real physical meaning or at least that our understanding of its properties, especially concerning its coupling to gravity, has to be clarified.

7 Conclusion

This short review aimed at presenting the main current ideas about the Casimir effect, its interpretation and its relevance to Cosmology. It does not reflect the increasing activity linked with Casimir-like effects in the nanoworld. See Ref. [25] for an introduction.

8 Appendix. Regularisation of Eq.23

As an example, we chose the regularisation provided by substitution (28) or

$$f(x) = \int_{x^2}^{\infty} dt \sqrt{t} e^{-\alpha t}. \tag{52}$$

The first three derivatives are given by

$$f'(x) = -2x^2 e^{-\alpha x^2}, \tag{53}$$

$$f''(x) = (-4x + 4\alpha x^3)e^{-\alpha x^2} \tag{54}$$

and

$$f'''(x) = (-4 + 20\alpha x^2 - 8\alpha^2 x^4)e^{-\alpha x^2}, \tag{55}$$

respectively. All derivatives at infinity are vanishing. It is also easy to verify that $f'(x)$ vanishes at $x = 0$ and that $f'''(0) = -4$ and that higher order derivatives at $x = 0$ are either vanishing or are proportional to positive powers of α. The first term in the curly braket of Eq. (14) being equal to $f(0)$, formula (27) results directly from the application of the Euler-McLaurin formula to the function (52), at the limit of vanishing α.

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