High-Energy QCD
Saturation and fluctuation effects

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Based on : G.S., hep-ph/0504129, Phys. Rev. D72:016007,2005

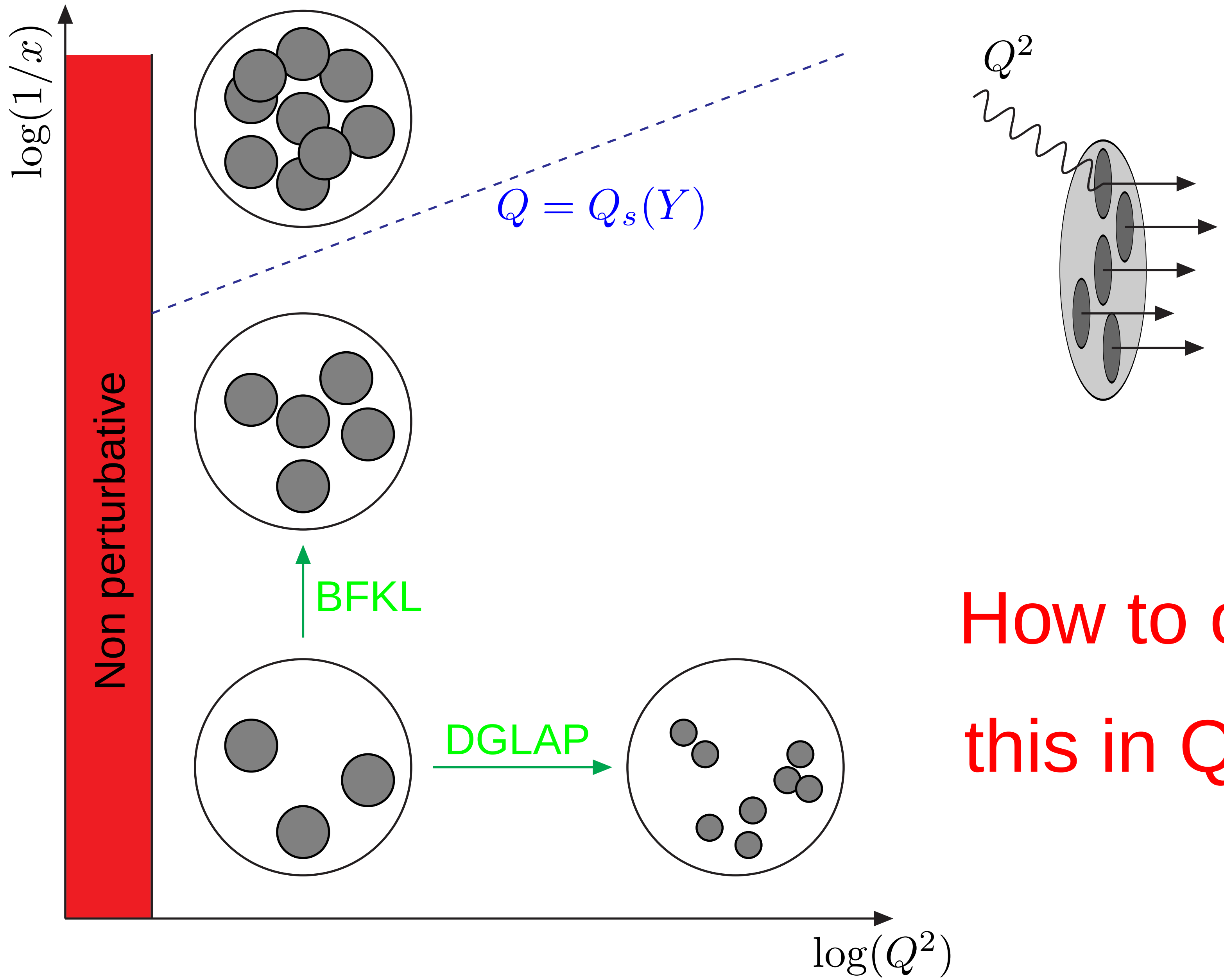
E. Iancu, G.S., D. Triantafyllopoulos, hep-ph/0510094, Nucl. Phys. A768 (2006) 194

Y. Hatta, E. Iancu, C. Marquet, G.S., D. Triantafyllopoulos, hep-ph/0601150

C. Marquet, R. Peschanski, G.S., hep-ph/0512186

Outline

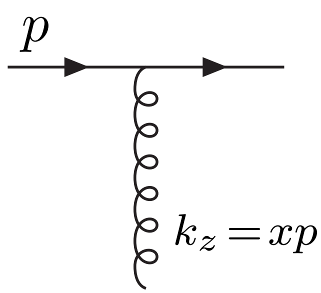
- Perturbative evolution in high-energy QCD:
  - Leading log approx.: Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation
  - Saturation effects: Balitsky-Kovchegov (BK) equation
  - Fluctation effects: new evolution as a reaction-diffusion process
- Asymptotic solutions: BK equation
  - Equivalence with statistical physics
  - Asymptotic properties: saturation scale and geometric scaling
- Asymptotic solution: including fluctuations
  - Stochastic evolution
  - Consequences: diffusive scaling
  - Present physical picture of high-energy QCD
- Outlook



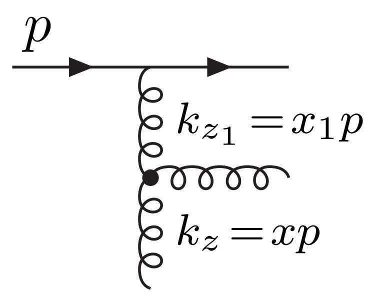
How to obtain this in QCD ?

Perturbative evolution in high-energy QCD

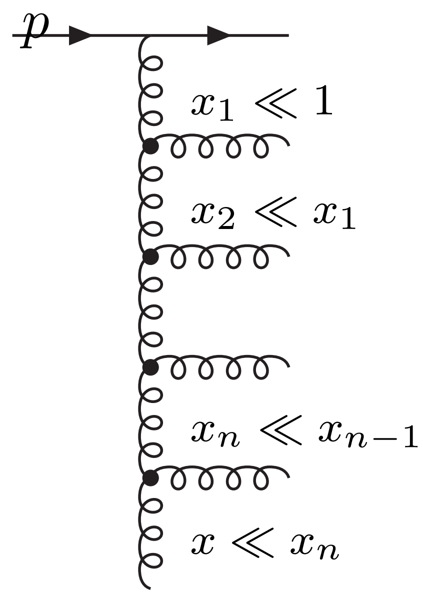
Bremsstrahlung:



$x \ll 1$



$x \ll x_1 \ll 1$



Probability of emission

$$dP \sim \alpha_s \frac{dk^2}{k^2} \frac{dx}{x}$$

In the small-x limit

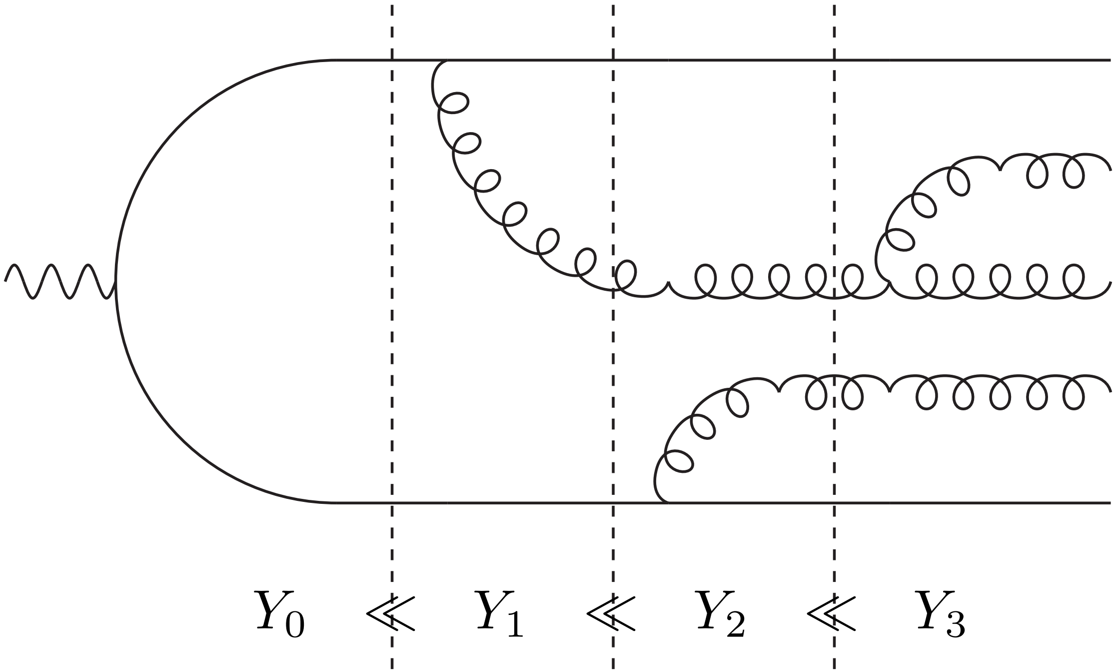
$$\int_x^1 \frac{dx_1}{x_1} \sim \alpha_s \log(1/x)$$

n-gluon emission $\longrightarrow \alpha_s^n \log^n(1/x)$

Dipole picture

Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

[Mueller,93]

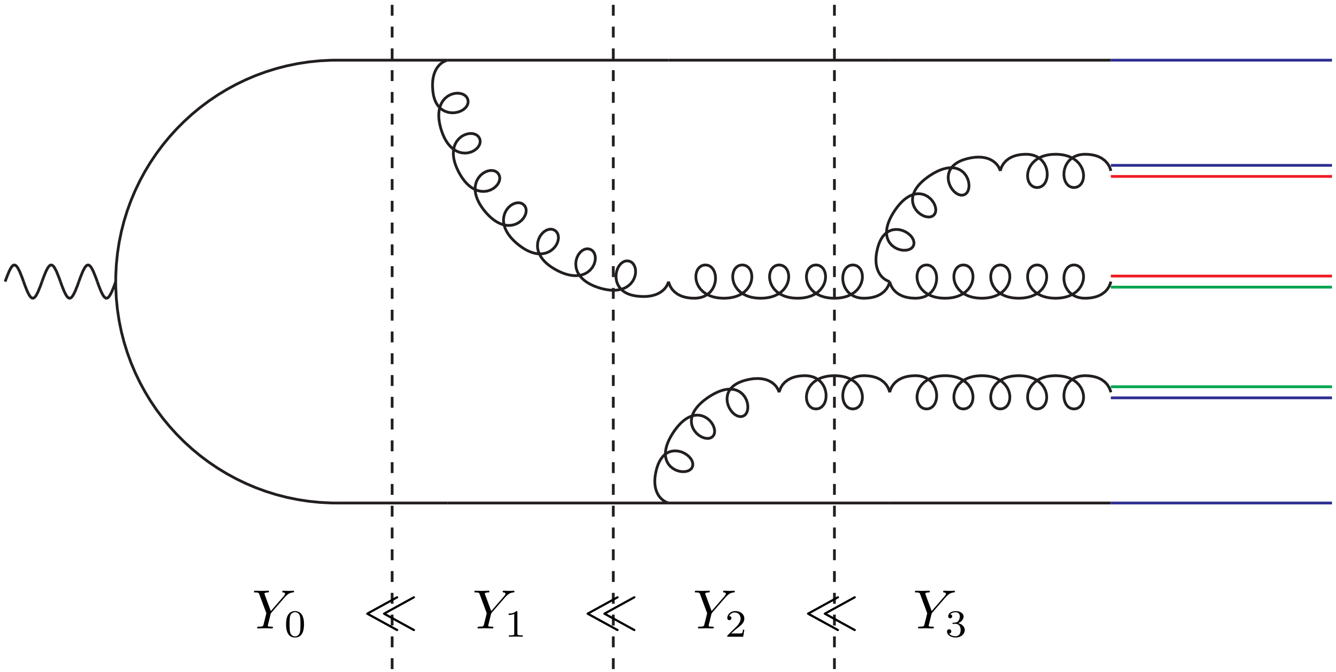


- Probability $\bar{\alpha}K$ of emission
- Independent emissions in coordinate space (transverse plane)

Dipole picture

Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

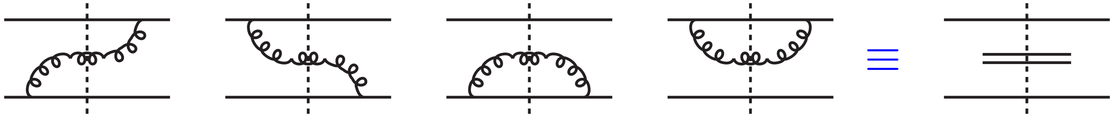
[Mueller,93]



$n(r, Y)$ dipoles

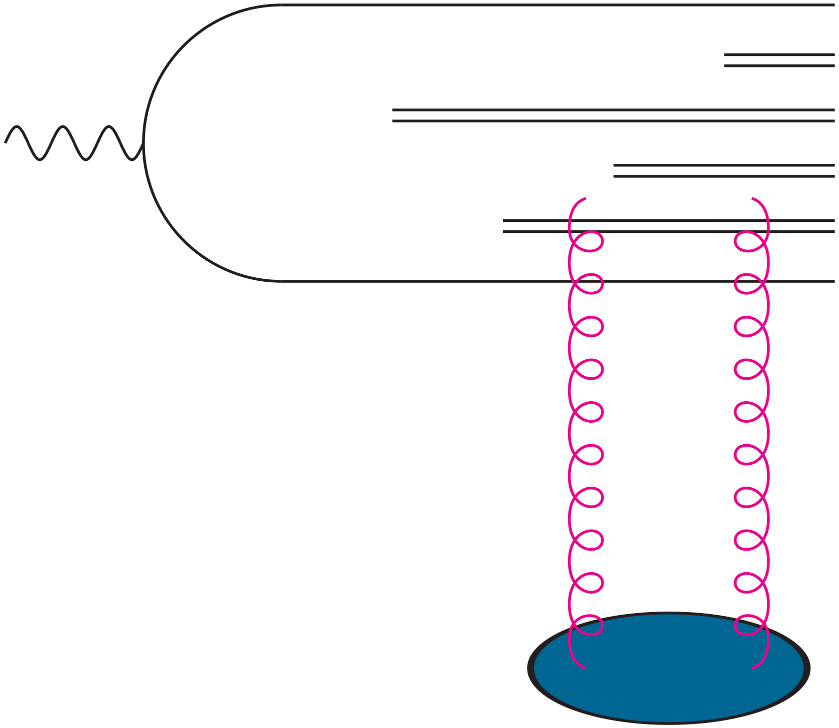
of size r

- Probability $\bar{\alpha}K$ of emission
- Independent emissions in coordinate space (transverse plane)
- Large-N_c approximation



BFKL evolution (2/4)

How to observe this system ?

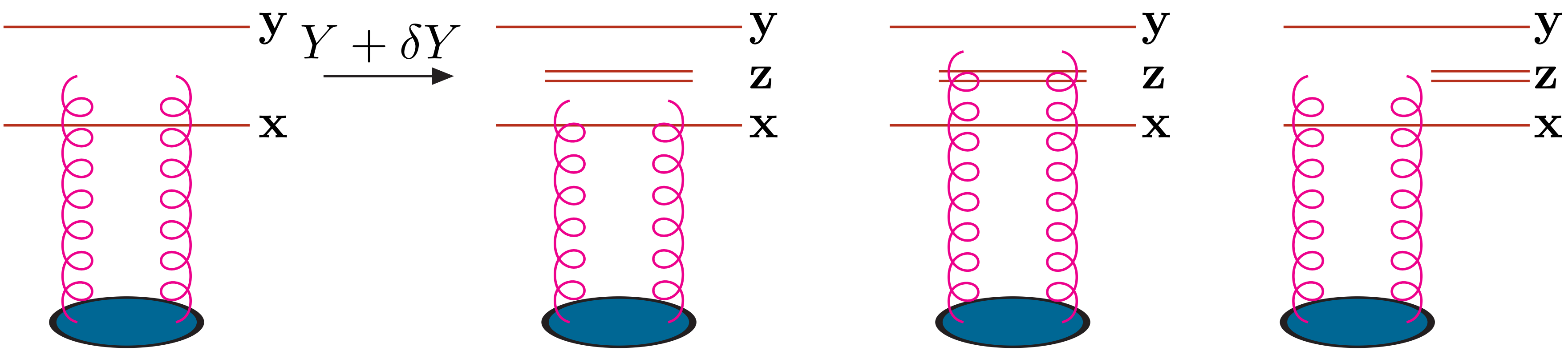


$$T(r, Y) \approx \alpha_s^2 n(r, Y)$$

Count the number of dipoles of a given size

BFKL evolution (3/4)

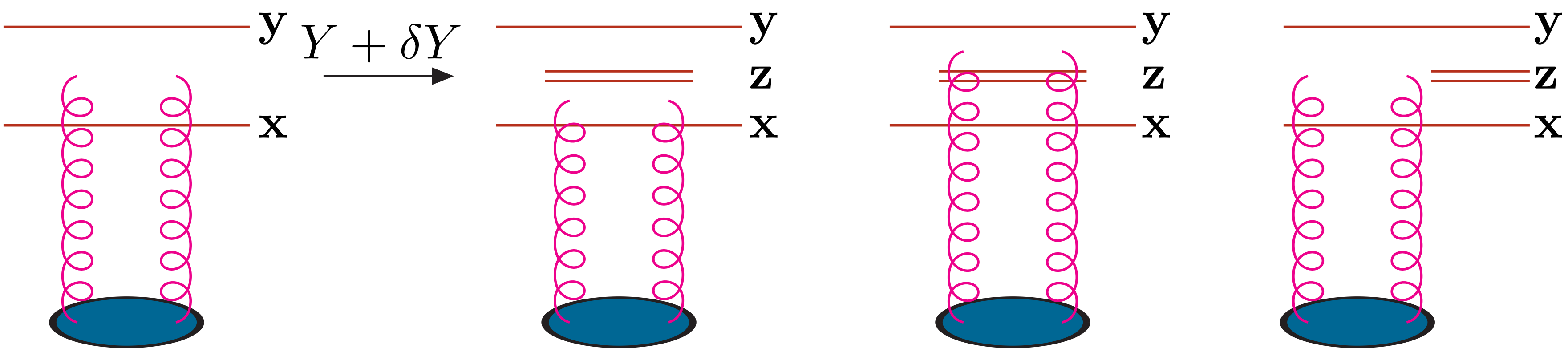
Consider a small increase in rapidity



$$\partial_Y T(\mathbf{x},\mathbf{y};Y)$$

$$T(\mathbf{x},\mathbf{z};Y)+T(\mathbf{z},\mathbf{y};Y)-T(\mathbf{x},\mathbf{y};Y)$$

Consider a small increase in rapidity ⇒ splitting



$$\partial_Y T(\mathbf{x},\mathbf{y};Y) = \bar{\alpha}\int d^2z\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[T(\mathbf{x},\mathbf{z};Y)+T(\mathbf{z},\mathbf{y};Y)-T(\mathbf{x},\mathbf{y};Y)\right]$$

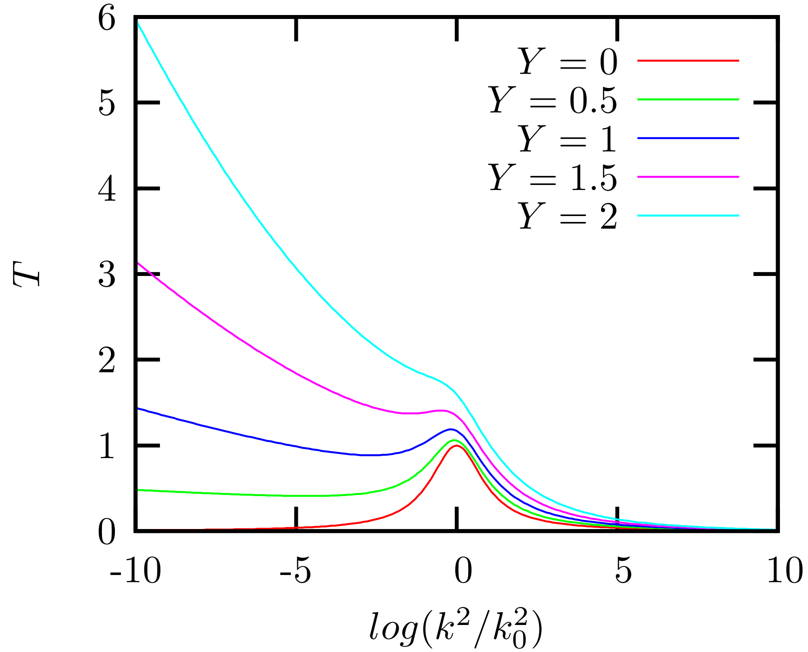
[Balitsky,Fadin,Kuraev,Lipatov,78]

BFKL evolution (4/4)

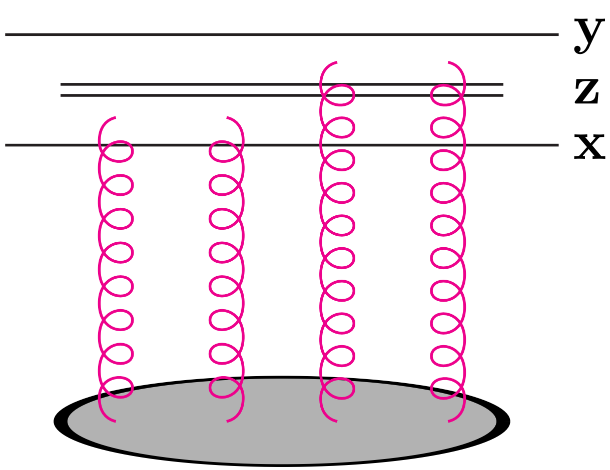
The solution goes like

$$T(Y) \sim e^{\omega Y} \qquad \text{with} \quad \omega = 4\bar{\alpha}\log(2) \approx 0.5$$

- Fast growth of the amplitude
- Intercept value too large
- Violation of the Froissart bound:
  $T(Y) \le C \log^2(s)$ $\qquad$ $T(r,b) \le 1$



+ problem of diffusion in the infrared



Multiple scattering

- ⋆ Proportional to T^2
- ⋆ important when $T \approx 1$

$$\partial_Y \langle T(\mathbf{x},\mathbf{y};Y)\rangle = \bar{\alpha}\int d^2z\, \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[\langle T(\mathbf{x},\mathbf{z};Y)\rangle + \langle T(\mathbf{z},\mathbf{y};Y)\rangle - \langle T(\mathbf{x},\mathbf{y};Y)\rangle - \langle T(\mathbf{x},\mathbf{z};Y)T(\mathbf{z},\mathbf{y};Y)\rangle\right]$$

contains

$$\partial\langle T(\mathbf{x},\mathbf{y};Y)\rangle \longrightarrow \langle T(\mathbf{x},\mathbf{z};Y)T(\mathbf{z},\mathbf{y};Y)\rangle$$

In general: complete hierarchy [Balitsky, 96]

$$\partial_Y \langle T^k \rangle \longrightarrow \underbrace{\langle T^k \rangle}_{\text{BFKL}}, \underbrace{\langle T^{k+1} \rangle}_{\text{saturation}}$$

Mean field approx.: $\langle T_{\mathbf{xz}} T_{\mathbf{zy}} \rangle = \langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle$

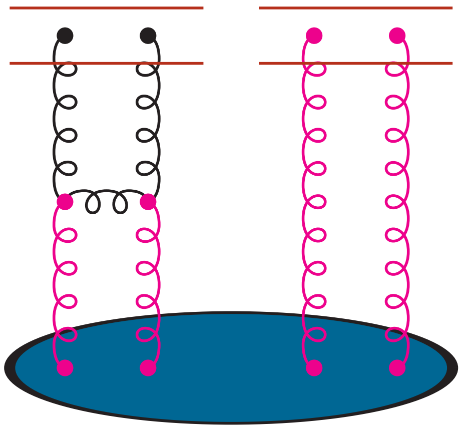
$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2z \, \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left[\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle - \langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle \right]$$

[Balitsky 96,Kovchegov 99]

Simplest perturbative evolution equation satisfying unitarity constraint

Consider evolution of $\langle T^{(2)} \rangle$

[E. Iancu, D. Triantafyllopoulos, 05]
Also A. Mueller, S. Munier, A. Shoshi, S. Wong



- Usual BFKL ladder

$$\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$$

Fluctuations

Consider evolution of $\langle T^{(2)} \rangle$

[E. Iancu, D. Triantafyllopoulos, 05]
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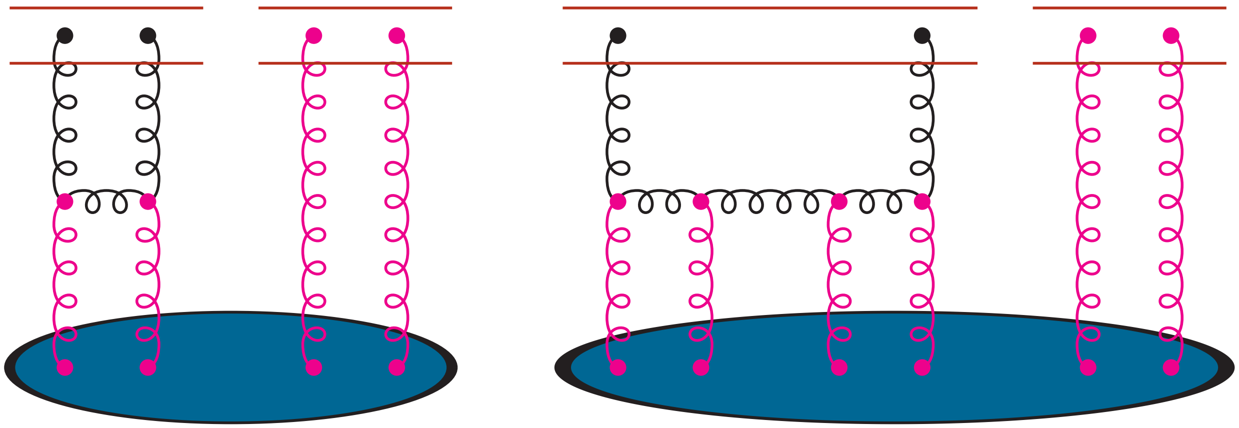

- Usual BFKL ladder $\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$
- fan diagram $\longrightarrow$ saturation effects $\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(3)} \rangle$

Consider evolution of $\langle T^{(2)} \rangle$

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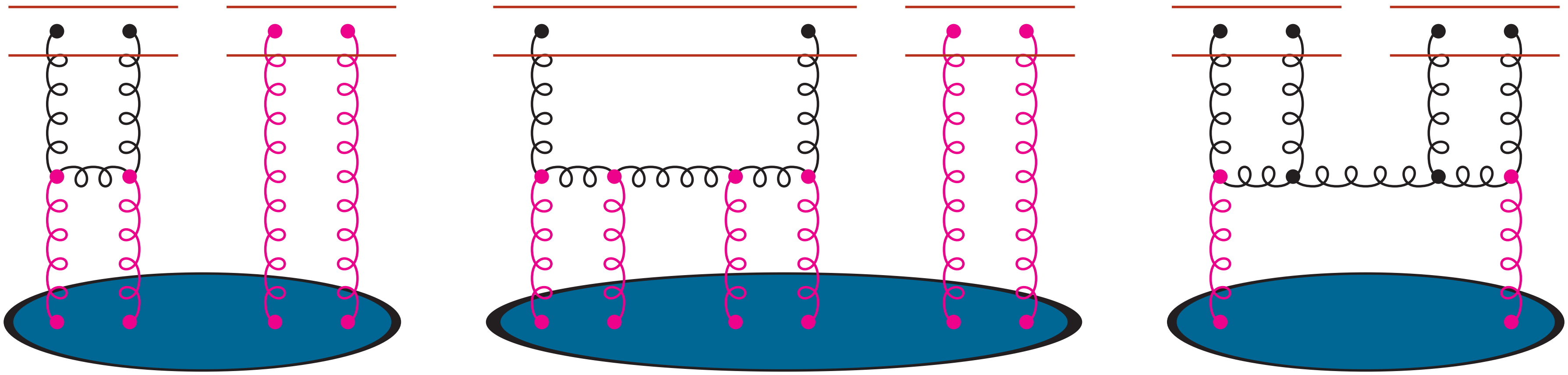


- Usual BFKL ladder $\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$
- fan diagram $\longrightarrow$ saturation effects $\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(3)} \rangle$
- splitting $\longrightarrow$ fluctuations, pomeron loops $\partial_Y \langle T^{(2)} \rangle \propto \langle T \rangle$

Evolution hierarchy

$\Rightarrow$ complicated hierarchy

$$\partial_Y \langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{y}_2)\rangle$$

$$= \frac{\bar\alpha}{2\pi}\int d^2z\,\frac{(\mathbf{x}_2-\mathbf{y}_2)^2}{(\mathbf{x}_2-\mathbf{z})^2(\mathbf{z}-\mathbf{y}_2)^2}\Big[\langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{z})\rangle + \langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{z},\mathbf{y}_2)\rangle$$

$$-\langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{y}_2)\rangle - \langle T^{(3)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{z};\mathbf{z},\mathbf{y}_2)\rangle + (1\leftrightarrow 2)\Big]$$

$$+ \frac{\bar\alpha}{2\pi}\left(\frac{\alpha_s}{2\pi}\right)^2\int_{\mathbf{uvz}} \mathcal{M}_{\mathbf{uvz}}\mathcal{A}_0(\mathbf{x}_1\mathbf{y}_1|\mathbf{uz})\mathcal{A}_0(\mathbf{x}_2\mathbf{y}_2|\mathbf{zv})\,\nabla^2_{\mathbf{u}}\nabla^2_{\mathbf{v}}\langle T^{(1)}(\mathbf{u},\mathbf{v})\rangle$$

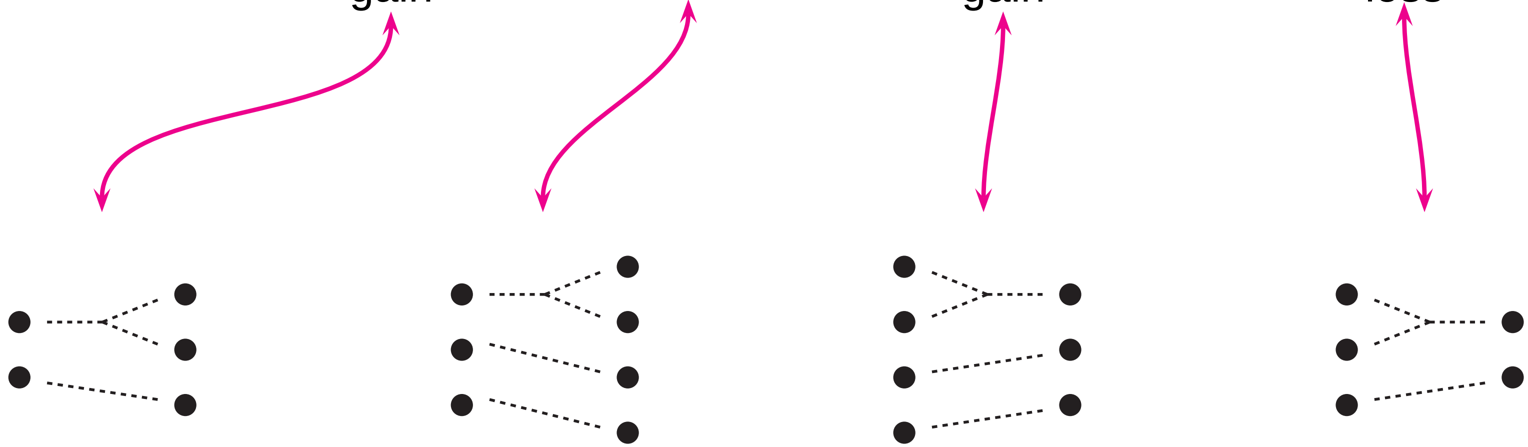
- Saturation: important when $T^{(2)} \sim T^{(1)} \sim 1$ i.e. near unitarity
- Fluctuations: important when $T^{(2)} \sim \alpha_s^2 T^{(1)}$ or $T \sim \alpha_s^2$ i.e. dilute regime

Reaction-diffusion

Reaction-diffusion process $A \underset{\sigma}{\overset{\gamma}{\rightleftharpoons}} A + A$

Master equation: $P_n \equiv$ proba to have n particles

$$\partial_t P_n = \underbrace{\gamma\,(n-1)P_{n-1}}_{\text{gain}} - \underbrace{\gamma\,nP_n}_{\text{loss}} + \underbrace{\sigma\,n(n+1)P_{n+1}}_{\text{gain}} - \underbrace{\sigma\,n(n-1)P_n}_{\text{loss}}$$



Particle densities: we observe a subset of k particles

$$\langle n^k \rangle \equiv \sum_{N=k}^{\infty} \frac{N!}{(N-k)!} P_N$$

Reaction-diffusion process $A \underset{\sigma}{\overset{\gamma}{\rightleftharpoons}} A + A$

Master equation: $P_n \equiv$ proba to have n particles

$$\partial_t P_n = \underbrace{\gamma\,(n-1)P_{n-1}}_{\text{gain}} - \underbrace{\gamma\,n P_n}_{\text{loss}} + \underbrace{\sigma\,n(n+1)P_{n+1}}_{\text{gain}} - \underbrace{\sigma\,n(n-1)P_n}_{\text{loss}}$$

Evolution equation: $\langle n^k \rangle \equiv$ particle density/correlators

$$\partial_t \langle n^k \rangle = \gamma\,k\langle n^k \rangle + \gamma\,k(k-1)\langle n^{k-1} \rangle - \sigma\,k(k+1)\langle n^{k+1} \rangle$$

Scattering amplitude for this system off a target

$$\mathcal{A}(t) = \sum_{k=0}^{\infty} (-)^k \, \langle n^k \rangle_{t_0} \, \langle T^k \rangle_{t-t_0}$$

t_0-independent $\Rightarrow$

$$\partial_t \langle T^k \rangle = \underbrace{\gamma \langle T^k \rangle}_{\text{BFKL}} - \underbrace{\gamma \langle T^{k+1} \rangle}_{\text{sat.}} + \underbrace{\sigma \langle T^{k-1} \rangle}_{\text{fluct.}}$$

For QCD particle = (effective) dipoles

Dipole plitting $\equiv$ BFKL kernel

$$\gamma \sim \bar\alpha \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}$$

Effective dipole merging

$$\sigma(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2 \to \mathbf{u}\mathbf{v})$$
$$\sim \bar\alpha\alpha_s^2\nabla_{\mathbf{u}}^2\nabla_{\mathbf{v}}^2 \left\{\mathcal{M}_{\mathbf{uvz}} \log^2\left[\frac{(\mathbf{x}_1-\mathbf{u})^2(\mathbf{y}_1-\mathbf{z})^2}{(\mathbf{x}_1-\mathbf{z})^2(\mathbf{y}_1-\mathbf{u})^2}\right] \log^2\left[\frac{(\mathbf{x}_2-\mathbf{v})^2(\mathbf{y}_2-\mathbf{z})^2}{(\mathbf{x}_2-\mathbf{z})^2(\mathbf{y}_2-\mathbf{v})^2}\right]\right\}$$

Remarks:

- merging not always positive
- fluctuations = gluon-number fluctuations
- Can be obtained from projectile or target point of view
- Known at large N_c.

Solutions

The BK equation

Link with statistical physics (1/3)

[S. Munier, R. Peschanski]

b-independent situation: momentum space ($L = \log(k^2/k_0^2)$)

$$\partial_{\bar{\alpha}Y} T(k) = \chi_{\text{BFKL}}(-\partial_L) T(k) - T^2(k)$$

Diffusive approximation:

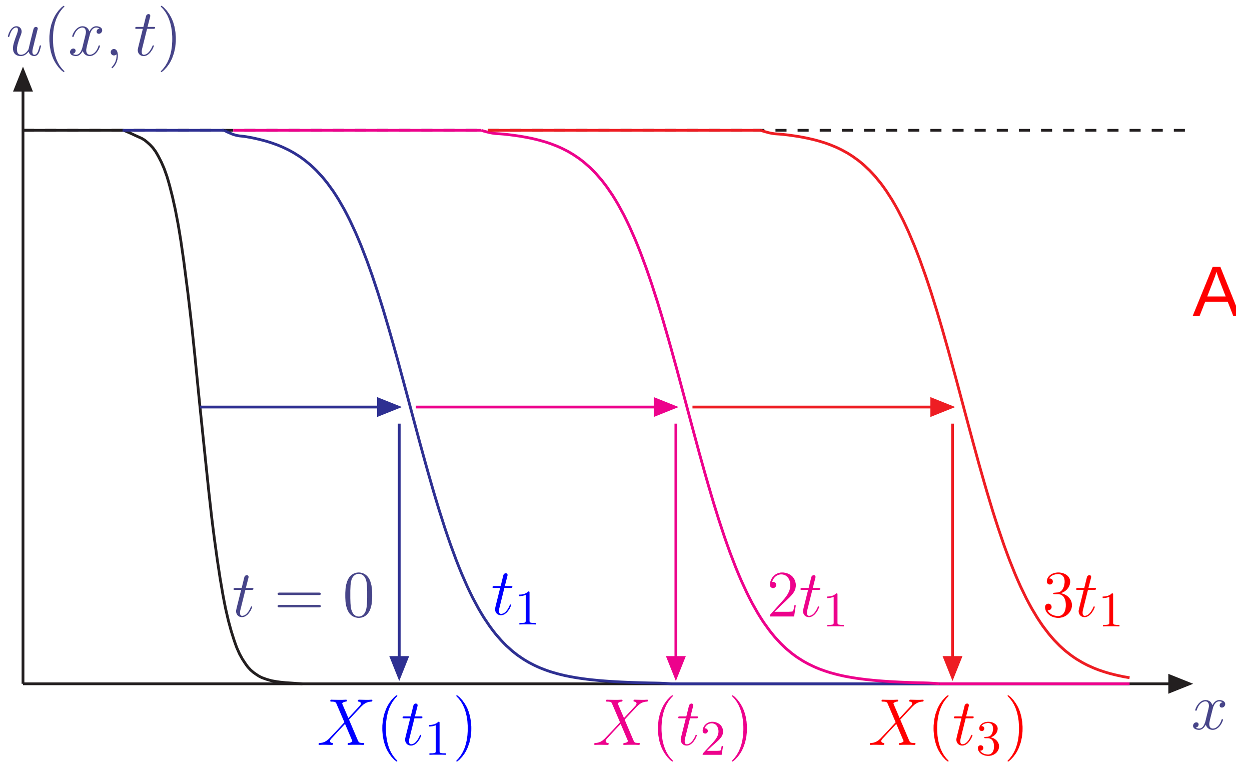
$$\chi_{\text{BFKL}}(-\partial_L) = \chi(\tfrac{1}{2}) + \tfrac{1}{2}(\partial_L + \tfrac{1}{2})^2$$

Time $t = \bar{\alpha}Y$, Space $x \approx \log(k^2)$, $u \propto T$

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u(x,t) - u^2(x,t)$$

Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP)

Link with statistical physics (2/3)



Asymptotic solution:

traveling wave

$$u(x,t) = u(x - v_c t)$$

Position: $X(t) = X_0 + v_c t$

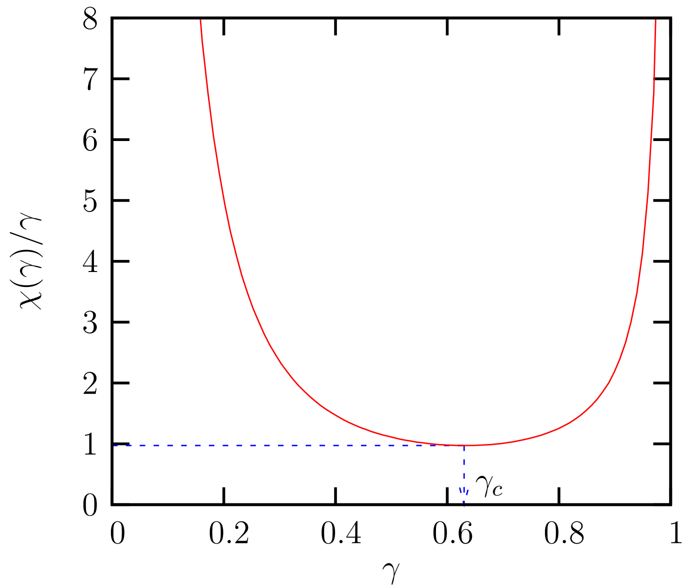
Link with statistical physics (2/3)

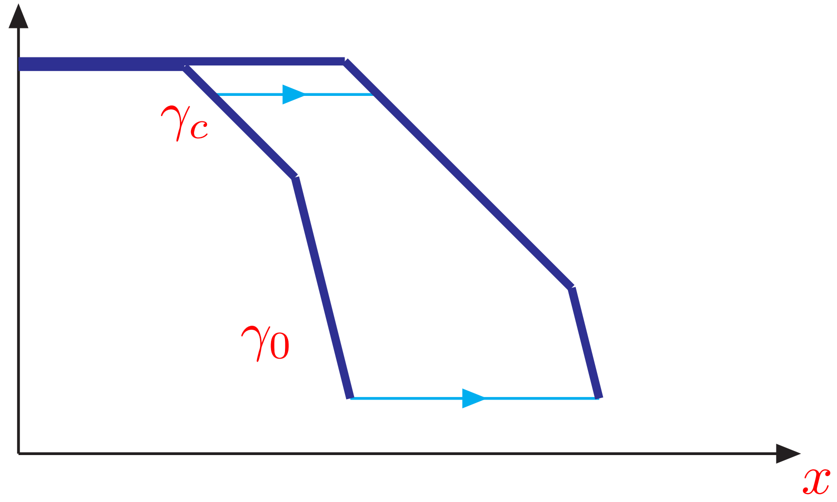
Mechnism: take only the linear part

$$\underbrace{\partial_Y T = \chi(-\partial_L)T}\, - T^2$$

$$T(k) = \int \frac{d\gamma}{2i\pi})\, T_0(\gamma)\, \exp\left(\chi(\gamma)Y - \gamma L\right)$$

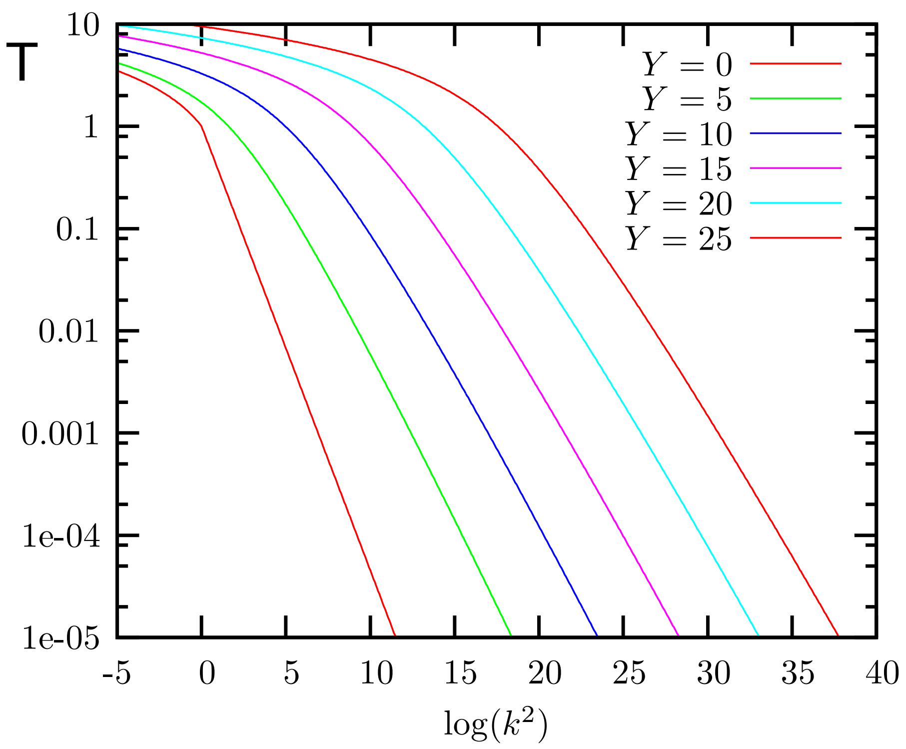
$\Rightarrow$ Wave of slope γ travels at speed $v = \chi(\gamma)/\gamma$

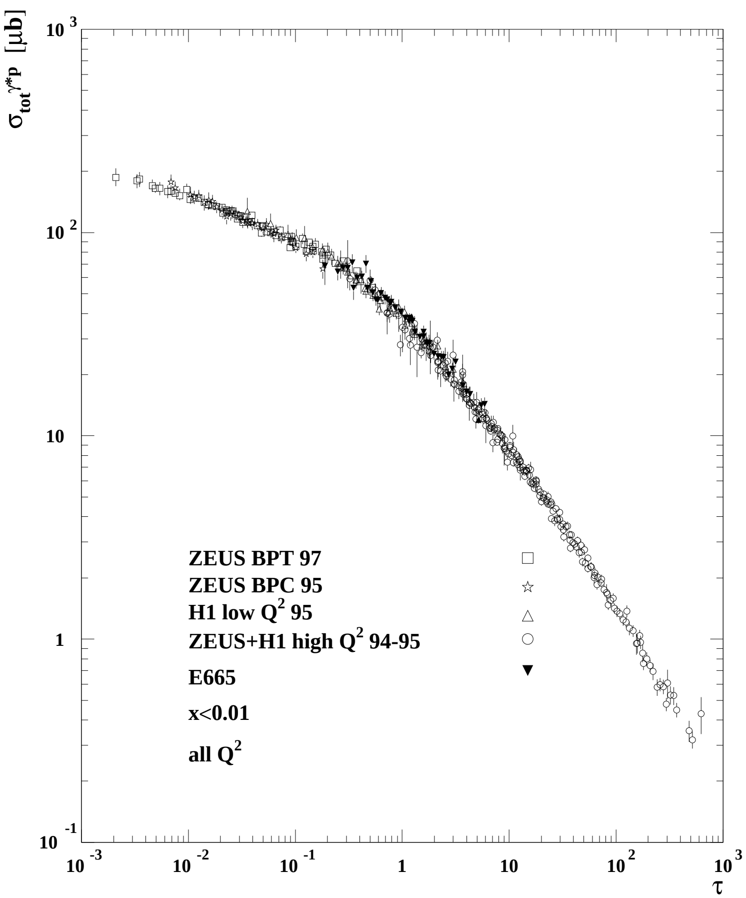




The minimal speed is selected during evoution

Numerical simulations:





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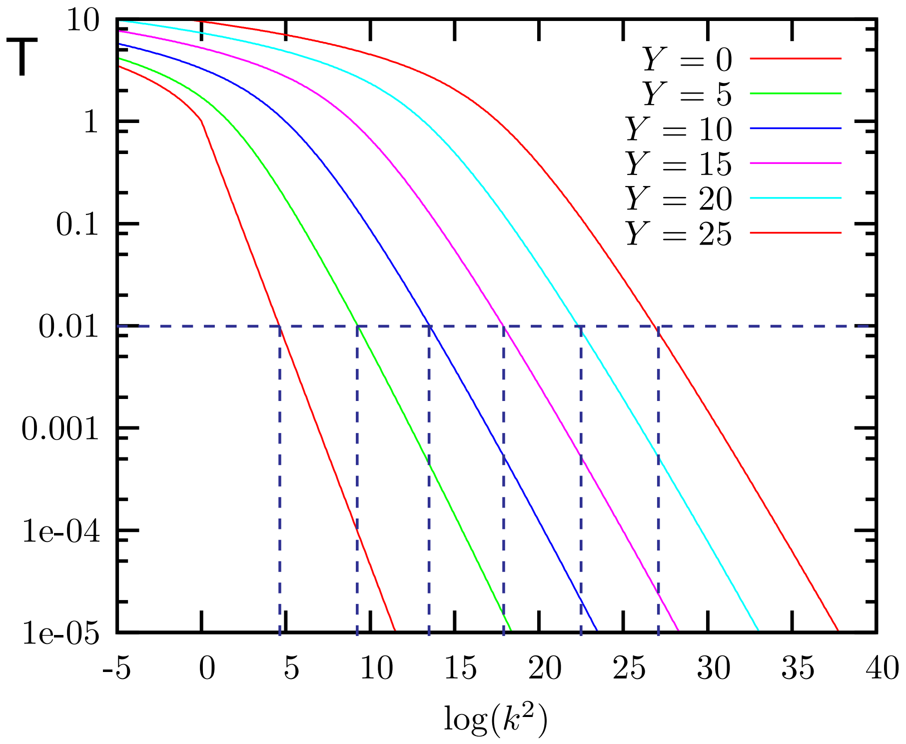
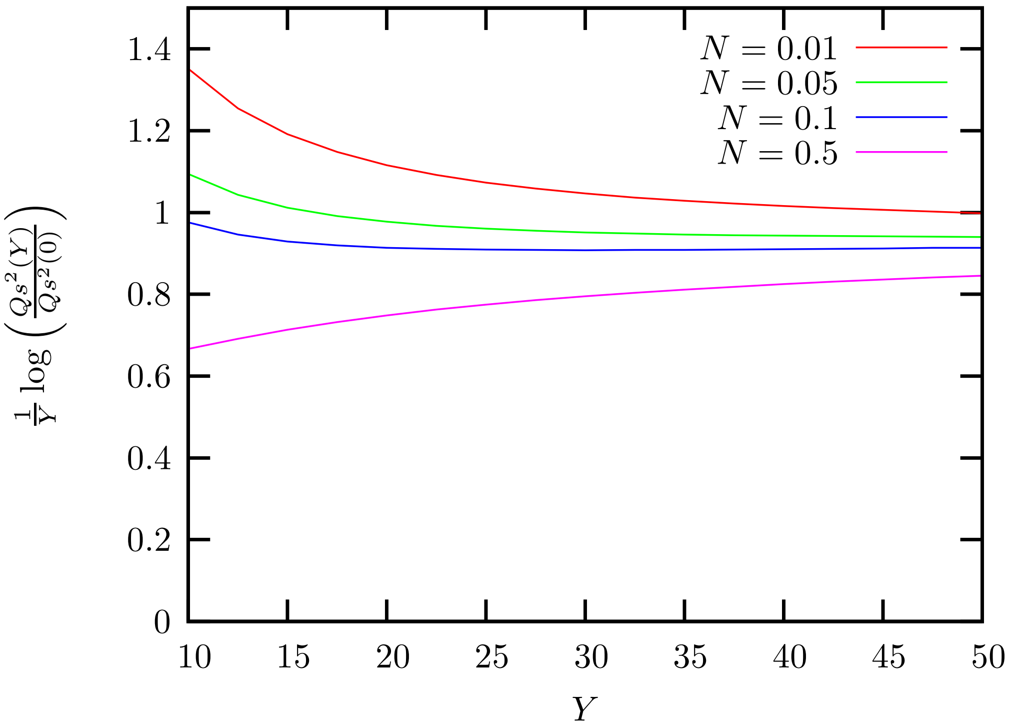




$$T(k,Y) = T\left(\frac{k^2}{Q_s^2(Y)}\right) \approx \left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c}$$

$$Q_s^2(Y) \propto \exp^{v_c Y}$$

Can we extend this including the b dependence

Go to momentum space

$$\tilde{T}(\mathbf{k},\mathbf{q}) = \int d^2x\, d^2y\, e^{i\mathbf{k}.\mathbf{x}} e^{i(\mathbf{q}-\mathbf{k}).\mathbf{y}} \frac{T(\mathbf{x},\mathbf{y})}{(\mathbf{x}-\mathbf{y})^2}$$

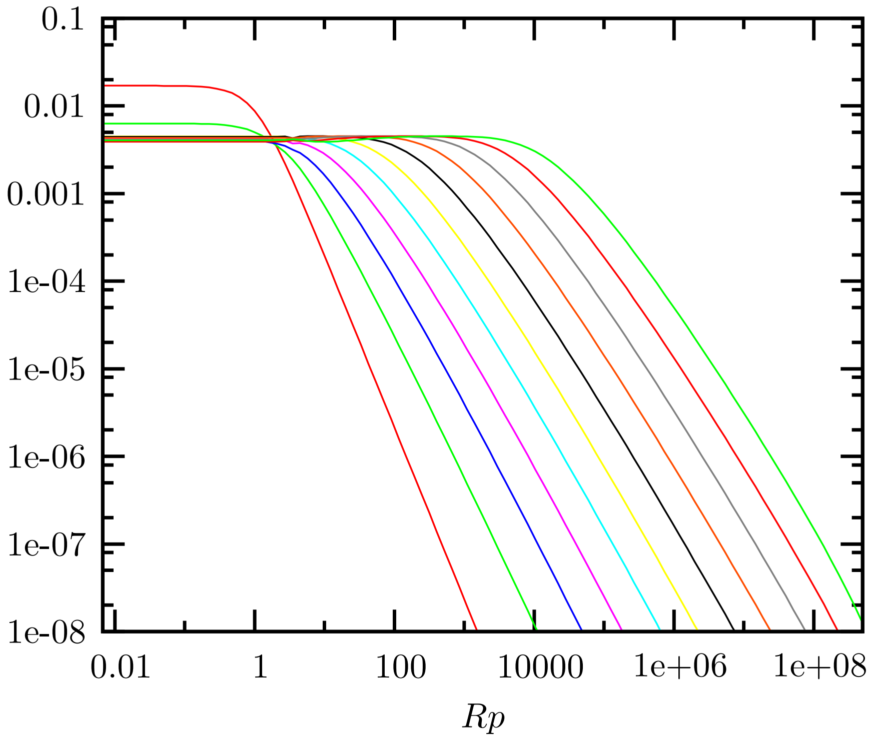
new form of the BK equation

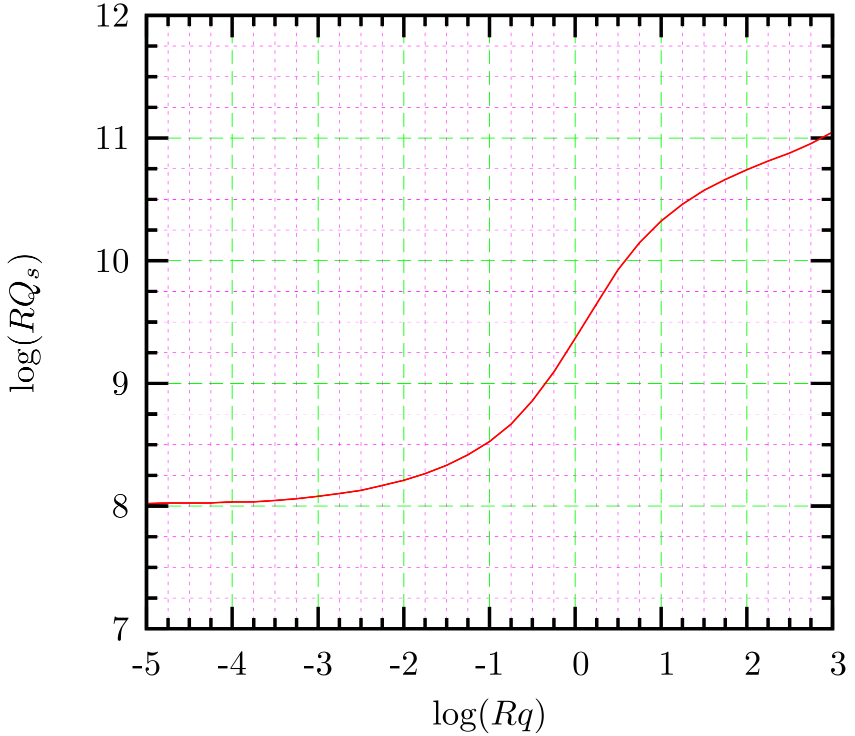
$$\begin{aligned}\partial_Y \tilde{T}(\mathbf{k},\mathbf{q}) &= \frac{\bar{\alpha}}{\pi}\int \frac{d^2k'}{(k-k')^2}\left\{\tilde{T}(\mathbf{k}',\mathbf{q}) - \frac{1}{4}\left[\frac{k^2}{k'^2} + \frac{(q-k)^2}{(q-k')^2}\right]\tilde{T}(\mathbf{k},\mathbf{q})\right\} \\ &- \frac{\bar{\alpha}}{2\pi}\int d^2k'\, \tilde{T}(\mathbf{k},\mathbf{k}')\tilde{T}(\mathbf{k}-\mathbf{k}',\mathbf{q}-\mathbf{k}')\end{aligned}$$

[C.Marquet, R.Peschanski, G.S., 05]

Numerical simulations

Dependence on momentum transfer k: traveling waves





One can prove analytically that:

- formation of a traveling wave at large p (or k)
- q dependence: scales like a constant or linearly ($Y = 25$)

Predicts geometric scaling for t-dependent processes

Solutions

Fluctuation effects

no b-dependence + coarse-graining $\longrightarrow$ Langevin equation

$$\partial_Y T(k,Y) = \bar{\alpha} K_{\text{BFKL}} \otimes T(k,Y) - \bar{\alpha} T^2(k,Y) + \bar{\alpha}\sqrt{\kappa\alpha_s^2 T(k,Y)}\,\nu(k,Y)$$

with $\langle \nu(k,Y) \rangle = 0$

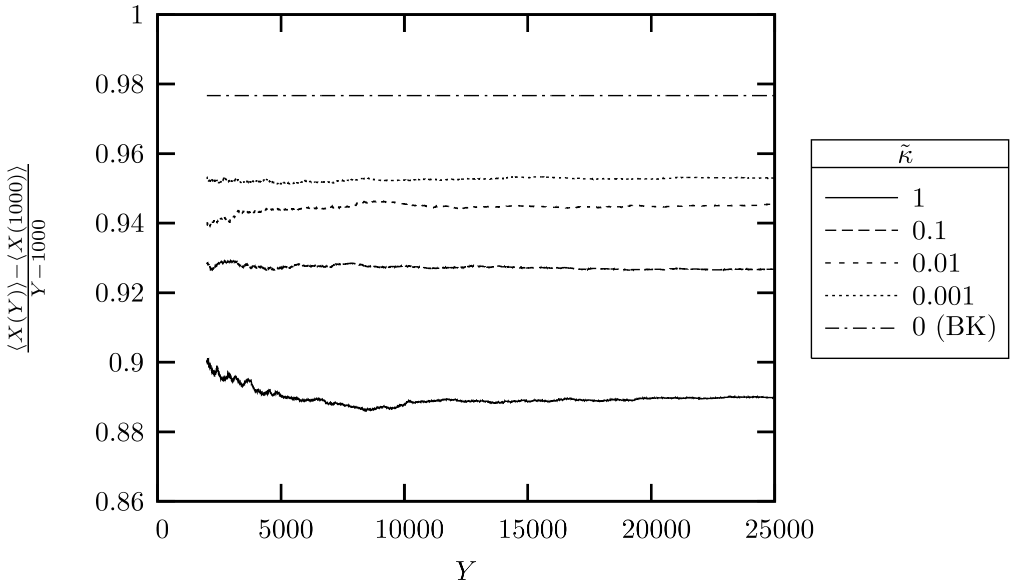
$$\langle \nu(k,Y)\nu(k',Y') \rangle = \frac{1}{\bar{\alpha}}\delta(Y-Y')\,k\delta(k-k')$$

Diffusive approximation

$$\partial_t u(x,t) = \partial_x^2 u + u - u^2 + \sqrt{2\kappa u}\,\nu(x,t)$$

stochastic F-KPP

[G.S., 05]

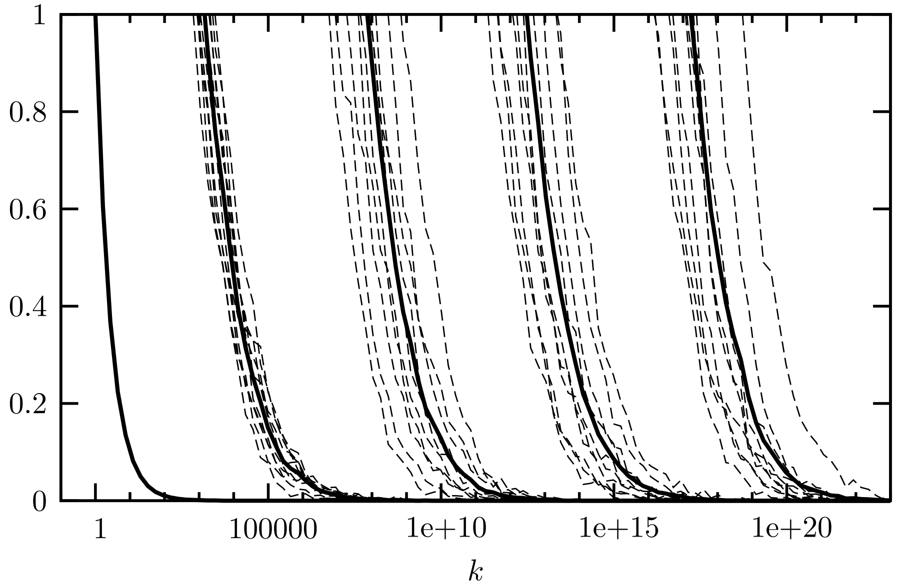


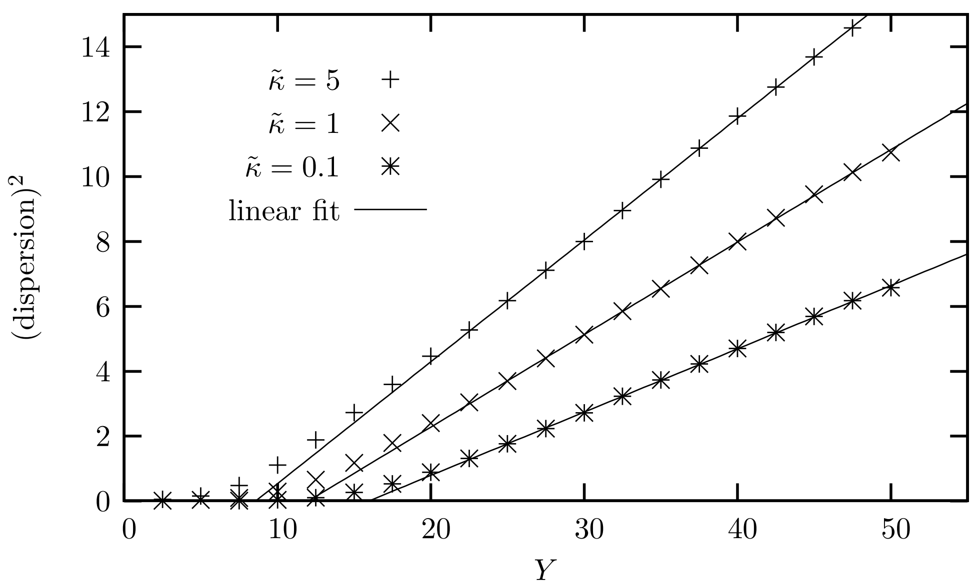
Decrease of the velocity/exponent of the saturation scale

For asymptotically small α_s (not true here)[A. Mueller, S. Munier, E. Brunet, B. Derrida]

$$v^* \underset{\alpha_s^2\kappa\to 0}{\longrightarrow} v_{BK} - \frac{\bar{\alpha}\pi^2\gamma_c\chi''(\gamma_c)}{2\log^2(\alpha_s^2\kappa)}$$

[G.S., 05]





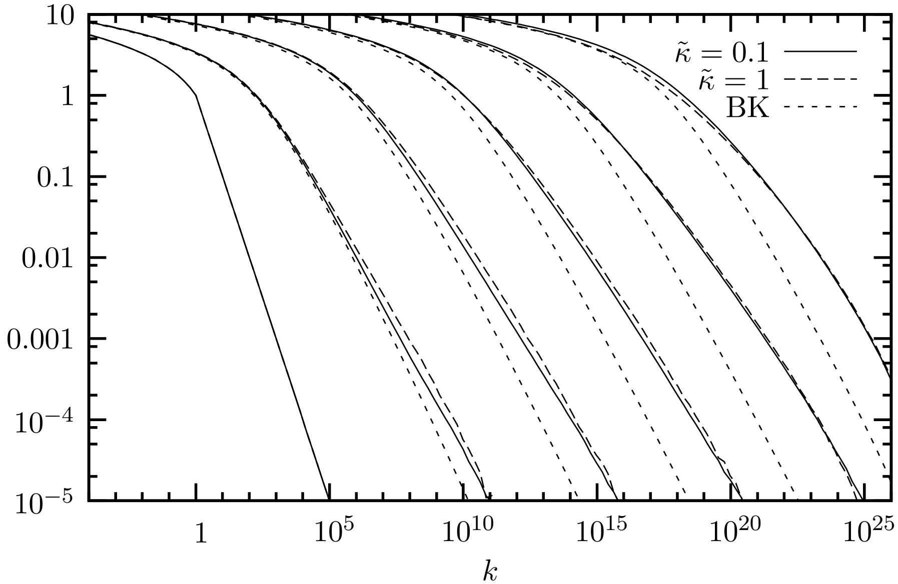
- Dispersion of the events

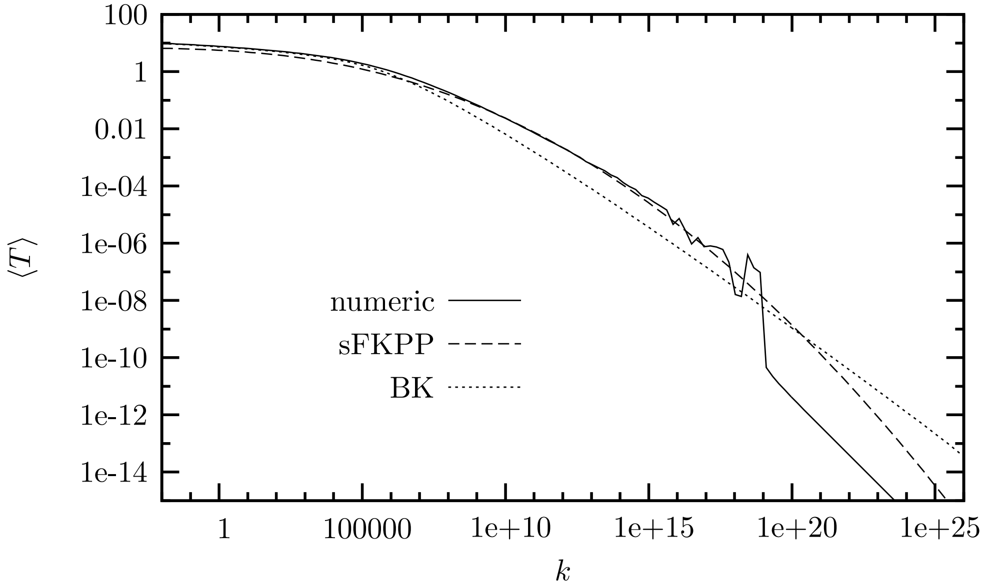
$$\Delta \log[Q_s^2(Y)] \approx \sqrt{D_{\text{diff}}\bar{\alpha} Y} \quad \text{with} \quad D_{\text{diff}} \underset{\alpha_s^2\kappa\to 0}{\sim} \frac{1}{|\log^3(\alpha_s^2\kappa)|}.$$

- No important dispersion in early stages of the evolution !

Averaged amplitude

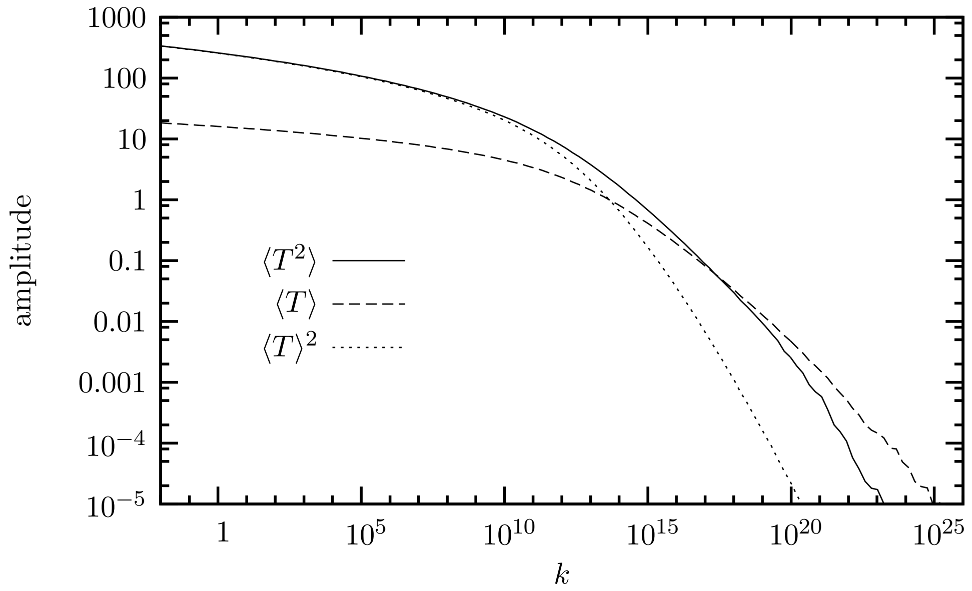
[G.S., 05]





- Clear effect of fluctuations
- Violations of geometric scaling (not in early stages)
- Agrees with predictions

[G.S., 05]



- Dense regime: $\langle T^2 \rangle \approx \langle T \rangle^2$
- Dilute regime: $\langle T^2 \rangle \approx \langle T \rangle$ (pre-asymptotics!)

High-energy behaviour

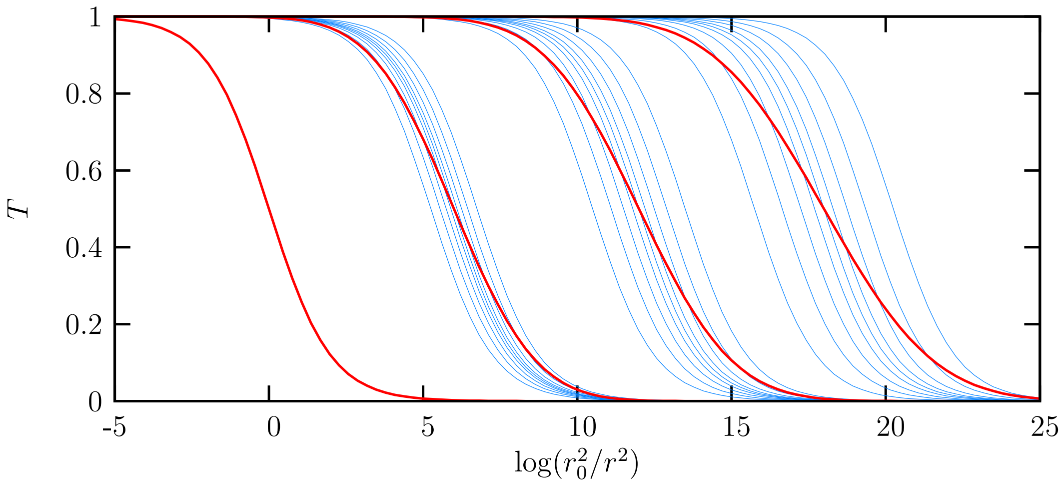
Evolution with saturation & fluctuations $\equiv$

- superposition of unitary front (with geometric scaling)
- with a dispersion (yielding geometric scaling violations)

$$\langle T(r,Y)\rangle = \int d\rho_s \, T_{\text{event}}(\rho-\rho_s)\, \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{(\rho_s-\bar{\rho}_s)^2}{\sigma^2}\right)$$

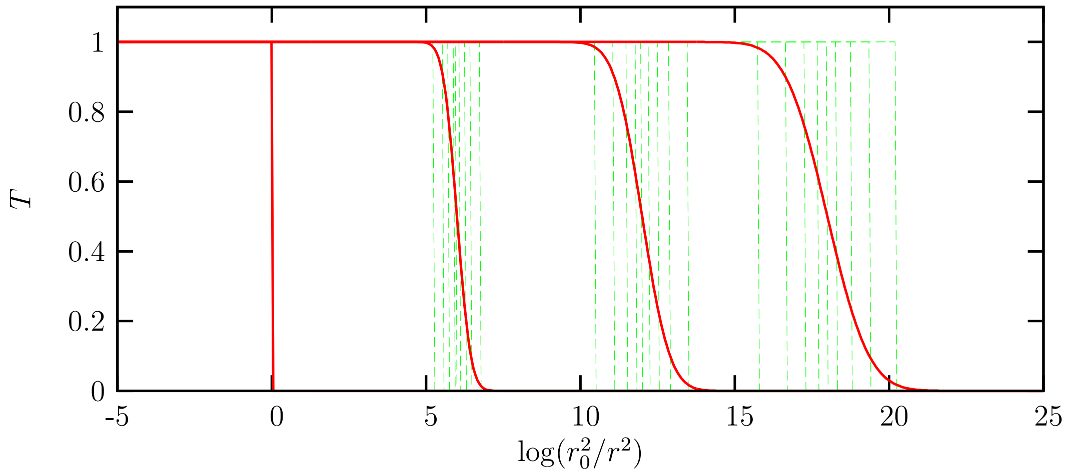
with $\rho = log(1/r^2)$, $\rho_s = log(Q_s^2)$

$$T_{\text{event}}(\rho-\rho_s) = \begin{cases} 1 & r > Q_s \\ (rQ_s)^\gamma & r < Q_s \end{cases}$$



dispersion $\sim DY$

- Y not too large $\Rightarrow$ small dispersion $\Rightarrow \langle T\rangle \approx T_{\text{event}} \Rightarrow$ geometric scaling
- Y very high $\Rightarrow$ dominated by disperion *i.e.* $\langle T\rangle \approx T_{\text{sat}}$



NB.: $\langle T^2\rangle = \langle T\rangle$

Strong noise limit

Intermediate energies	High energies
Mean field (BK)	Fluctuations
Geometric scaling $\langle T \rangle = f\left[\log(k^2/Q_s^2)\right]$ $\langle T^{(k)} \rangle = \langle T \rangle^k$	Diffusive scaling $\langle T \rangle = f\left[\log(k^2/Q_s^2)/\sqrt{DY}\right]$ $\langle T^{(k)} \rangle = \langle T \rangle$

At high-energy, amplitudes are dominated by hot-spots *i.e.* rare fluctuations at saturation

- true for strong fluctuations
- asymptotically true in general

Saturation fit:

$$\langle T(r,Y)\rangle = \begin{cases} (r^2Q_s^2)^{\gamma_c} e^{-\frac{2\log^2(rQ_s)}{CY}} & r < Q_s \\ 1 - e^{-a-b\log^2(rQ_s)} & r > Q_s \end{cases}$$

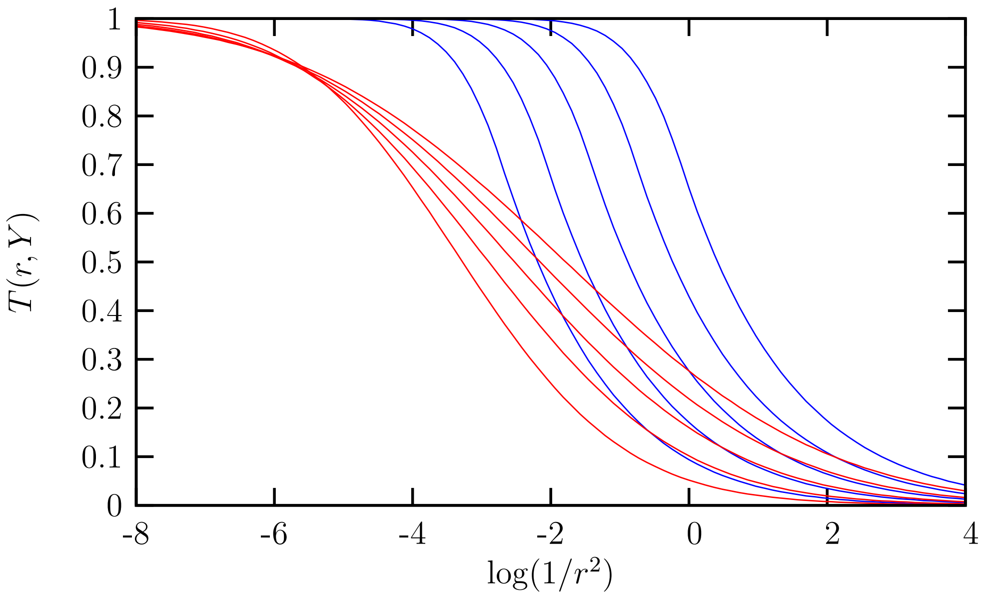
$$Q_s^2(Y) = \lambda Y,\ \rho_s = \log(Q_s^2)$$

Saturation+fluctuations fit:

$$\langle T(r,Y)\rangle = \int d\rho_s\, T(r,\rho_s) \frac{1}{\sqrt{\pi}\sigma} e^{-\frac{(\rho_s-\bar\rho_s)^2}{\sigma^2}}$$

$$T(r,\rho_s) = \begin{cases} r^2Q_s^2 & r < Q_s \\ 1 & r > Q_s \end{cases}$$

colour transparency

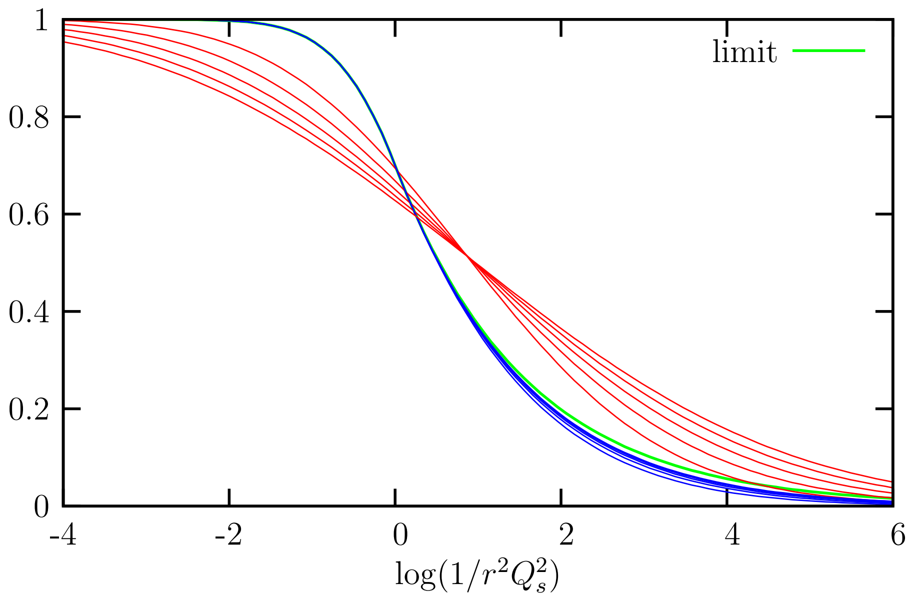


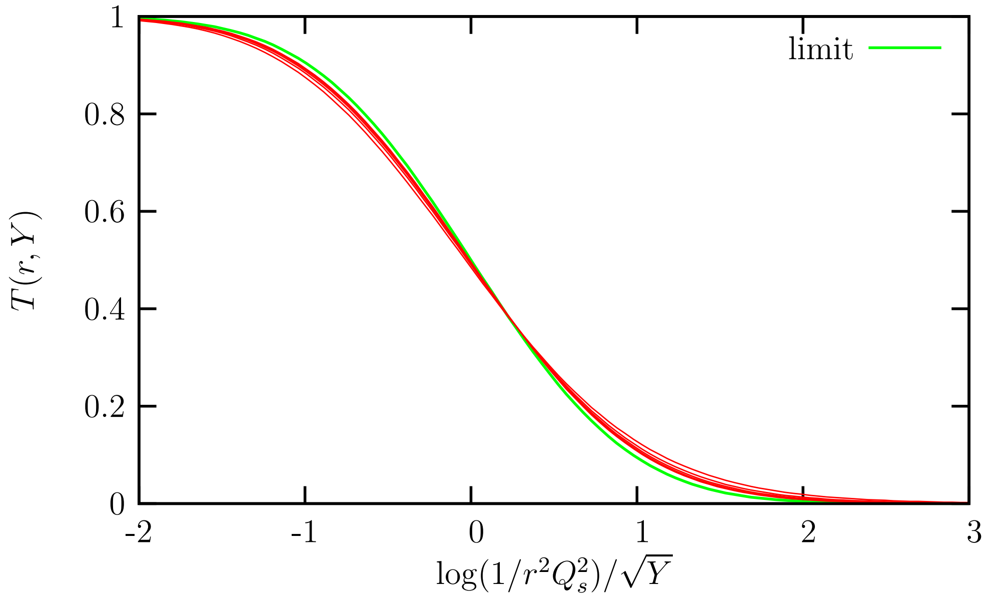
Saturation fit:

$$\langle T(r,Y)\rangle = (r^2Q_s^2)^{\gamma_c} e^{-\frac{2\log^2(rQ_s)}{CY}} \qquad \to r^2Q_s^2$$

Saturation+fluctuations fit:

$$\langle T(r,Y)\rangle = \int d\rho_s\, T(r,\rho_s)\frac{1}{\sqrt{\pi}\sigma}e^{-\frac{(\rho_s-\bar\rho_s)^2}{\sigma^2}} \qquad \to \frac{1}{2}\text{erfc}\left(\frac{-\log(r^2\bar Q_s^2)}{\sqrt{2DY}}\right)$$

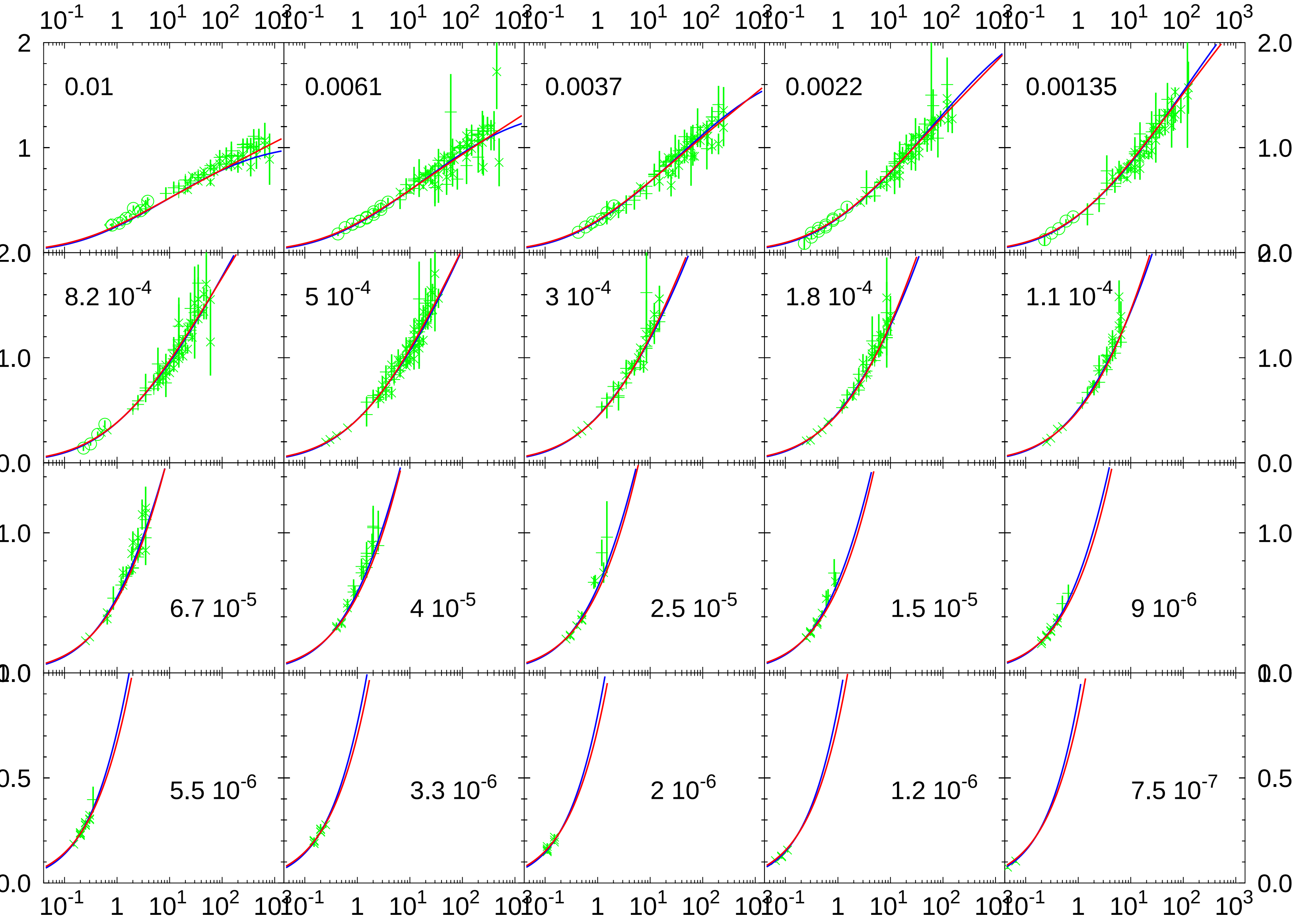




$\xrightarrow{Y\to\infty}$ Geometric scaling

$\xrightarrow{Y\to\infty}$ Diffusive scaling

Describing F_2



Both fits can describe the data for $x \leq 0.01$

Conclusion

- Effects of saturation
  - Evolution equations for high-energy QCD
    Balitsky, Balitsky-Kovchegov (dipole), JIMWLK (CGC)
  - Good knowledge of the asymptotic solutions
    Traveling waves $\to$ geometric scaling, saturation scale $\propto \exp(\bar{\alpha} v_c Y)$
- Effects of fluctuations
  - Known at large-N_c
  - Consequences on saturation (*e.g.* geometric scaling violations)
    Diffusive scaling
  - analytical solutions: $\alpha_s \lll 1$
    numerical solutions: coherent with statistical-physics analog

Perspectives

- phenomenological tests:
  - do we observe geometric scaling at nonzero momentum transfer ?
  - predictions for LHC ? diffusive scaling at high-energy ?

Perspectives

- phenomenological tests:
  - do we observe geometric scaling at nonzero momentum transfer ?
  - predictions for LHC ? diffusive scaling at high-energy ?
- theoretical questions:
  - importance of geometric scaling violations
  - analytical predictions (pomeron loops, triple pomeron vertex)
  - numerical simumations: include impact parameter
  - beyond large-N_c