Workshop HLPW 2008

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Integral equation for gauge invariant quark Green's function

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Objective

Investigate the possibilities of deriving integral or integro-differential equations for gauge invariant Green's functions. Those involve path-ordered gluon field phase factors. Here, we concentrate on two-point quark Green's functions, in which the path-ordered phase factor is made of a single straight line or more generally of a skew-polygonal line.

The starting point is a particular representation for the quark propagator in the presence of an external gluon field, where it is expressed as a series of terms involving path-ordered phase factors along successive straight lines. Then the corresponding quantized Green's function becomes expressed in terms of Wilson loops having skew-polygonal contours.

Definitions and conventions

Path-ordered gluon field phase factor along a line C_{yx} joining a point x to a point y, with an orientation defined from x to y:

$$U(C_{yx};y,x) \equiv U(y,x) = Pe^{-ig\int_x^y dz^{\mu}A_{\mu}(z)}.$$

Parametrizing the line C with a parameter λ, $0 \leq \lambda \leq 1$, such that $x(0) = x$ and $x(1) = y$, a variation of C induces the following variation of U (Mandelstam, 1968):

$$\begin{aligned}\delta U(1,0) &= -ig\delta x^{\alpha}(1)A_{\alpha}(1)U(1,0) + igU(1,0)A_{\alpha}(0)\delta x^{\alpha}(0) \\ &\quad +ig\int_0^1 d\lambda U(1,\lambda)x'^{\beta}(\lambda)F_{\beta\alpha}(\lambda)\delta x^{\alpha}(\lambda)U(\lambda,0),\end{aligned}$$

where $x' = \frac{\partial x}{\partial \lambda}$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig[A_{\mu},A_{\nu}]$.

For paths defined along rigid lines, the variations inside the integral are related, with appropriate weight factors, to those of the end points.

Considering now a rigid straight line between x and y, a derivation at the end points yields:

$$\frac{\partial U(y,x)}{\partial y^\alpha} = -igA_\alpha(y)U(y,x) + ig(y-x)^\beta \int_0^1 d\lambda\, \lambda\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\partial U(y,x)}{\partial x^\alpha} = +igU(y,x)A_\alpha(x) + ig(y-x)^\beta \int_0^1 d\lambda\, (1-\lambda)\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Conventions to represent the contributions of the integrals:

$$\frac{\bar{\delta}U(y,x)}{\bar{\delta}y^{\alpha+}} \equiv ig(y-x)^\beta \int_0^1 d\lambda\, \lambda\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\bar{\delta}U(y,x)}{\bar{\delta}x^{\alpha-}} \equiv ig(y-x)^\beta \int_0^1 d\lambda\, (1-\lambda)\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Wilson loop

$$\Phi(C) = \frac{1}{N_c} \mathrm{tr} P e^{-ig \oint_C dx^\mu A_\mu(x)}.$$

Vacuum expectation value:

$$W(C) = \langle \Phi(C) \rangle.$$

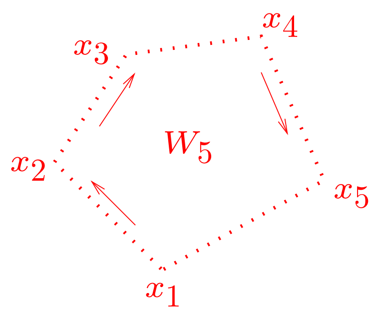
Functional representation:

$$W(C) = e^{F(C)}.$$

In perturbation theory, $F(C)$ is given by the sum of all connected diagrams, the connection being defined with respect to the contour C (Dotsenko and Vergeles, 1980). For large contours and large N_c, $F(C)$ is proportional to the minimal surface with contour C (Makeenko and Migdal, 1980).

If the contour C is a skew-polygon C_n with n sides and n successive marked points x_1, x_2, ..., x_n at the cusps, then we write:

$$W(x_n, x_{n-1}, \ldots, x_1) = W_n = e^{F_n(x_n, x_{n-1}, \ldots, x_1)} = e^{F_n}.$$



$$W_5(x_5, x_4, \ldots, x_1) = e^{F_5(x_5, \ldots, x_1)}$$

Two-point Green's functions

The gauge invariant two-point quark Green's function is defined as

$$S_{\alpha\beta}(x,x';C_{x'x}) = -\frac{1}{N_c}\,\langle\overline{\psi}_\beta(x')\,U(C_{x'x};x',x)\,\psi_\alpha(x)\rangle.$$

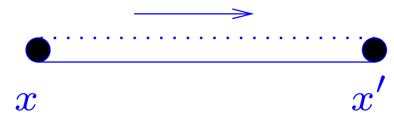
For skew-polygonal lines with n sides and $n-1$ junction points y_1, y_2, $\ldots$,y_{n-1} between the segments, we define:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = -\frac{1}{N_c}\,\langle\overline{\psi}(x')U(x',y_{n-1})U(y_{n-1},y_{n-2})\ldots U(y_1,x)\psi(x)\rangle.$$

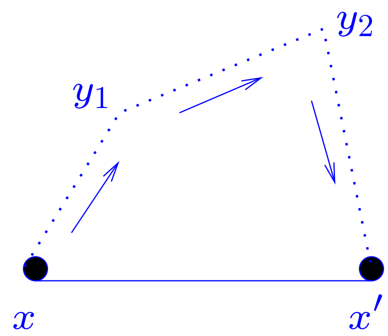
For one straight line, one has:

$$S_{(1)}(x,x') \equiv S(x,x') = -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,U(x',x)\,\psi(x)\rangle.$$

Pictorially:



$$S(x,x') \equiv S_{(1)}(x,x') = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',x)\, \psi(x) >$$



$$S_{(3)}(x,x';y_2,y_1) = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',y_2)U(y_2,y_1)U(y_1,x)\, \psi(x) >$$

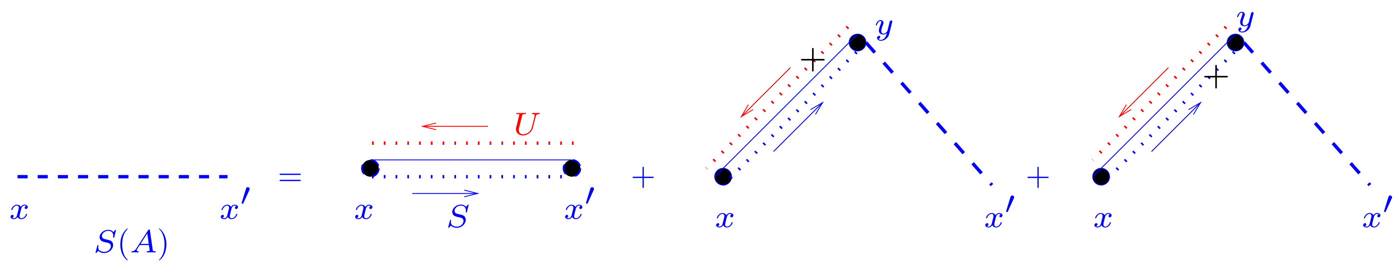
Quark propagator in the external gluon field

A two-step quantization. One first integrates with respect to the quark fields. This produces in various terms the quark propagator in the presence of the gluon field. Then one integrates with respect to the gluon field through Wilson loops.

To make Wilson loops appear, one needs an appropriate representation for the quark propagator in extenal field. We use the following representation which involves phase factors along straight lines together with the full quark Green's function $S_{(1)} \equiv S$ (F. Jugeau and H.S., 2003). Generalization of a representation introduced by Eichten and Feinberg, 1981, for heavy quarks.

$$S(x,x';A) = S(x,x')U(x,x') + \left(S(x,y)\frac{\bar{\delta}U(x,y)}{\bar{\delta}y^{\alpha-}} + \frac{\bar{\delta}S(x,y)}{\bar{\delta}y^{\alpha+}}U(x,y) \right)\gamma^{\alpha}S(y,x';A).$$

Pictorially:



This yields an expansion of $S(A)$ in terms of the gauge invariant Green's function S and explicit phase factors along straight lines.

Functional relations for Green's functions

Systematic use of the expansion of the quark propagator in external field.

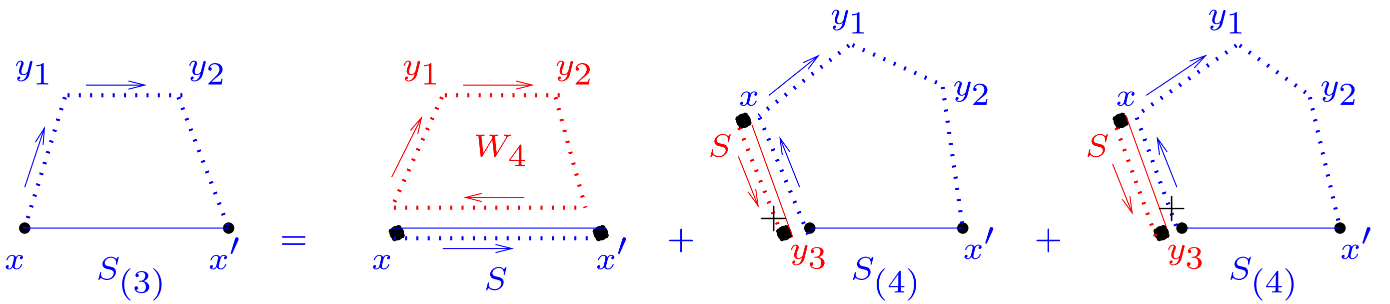
Consider the Green's function $S_{(n)}$. Integrate with respect to the quark fields:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = \frac{1}{N_c}\langle U(x',y_{n-1})U(y_{n-1},y_{n-2})\cdots U(y_1,x)S(x,x';A)\rangle.$$

Use of the expansion for $S(A)$ gives:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = S(x,x')\,e^{F_{n+1}(x',y_{n-1},\ldots,y_1,x)}$$
$$+\left(\frac{\bar{\delta}S(x,y_n)}{\bar{\delta}y_n^{\alpha+}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta}y_n^{\alpha-}}\right)\gamma^\alpha\,S_{(n+1)}(y_n,x';y_{n-1},\ldots,y_1,x).$$

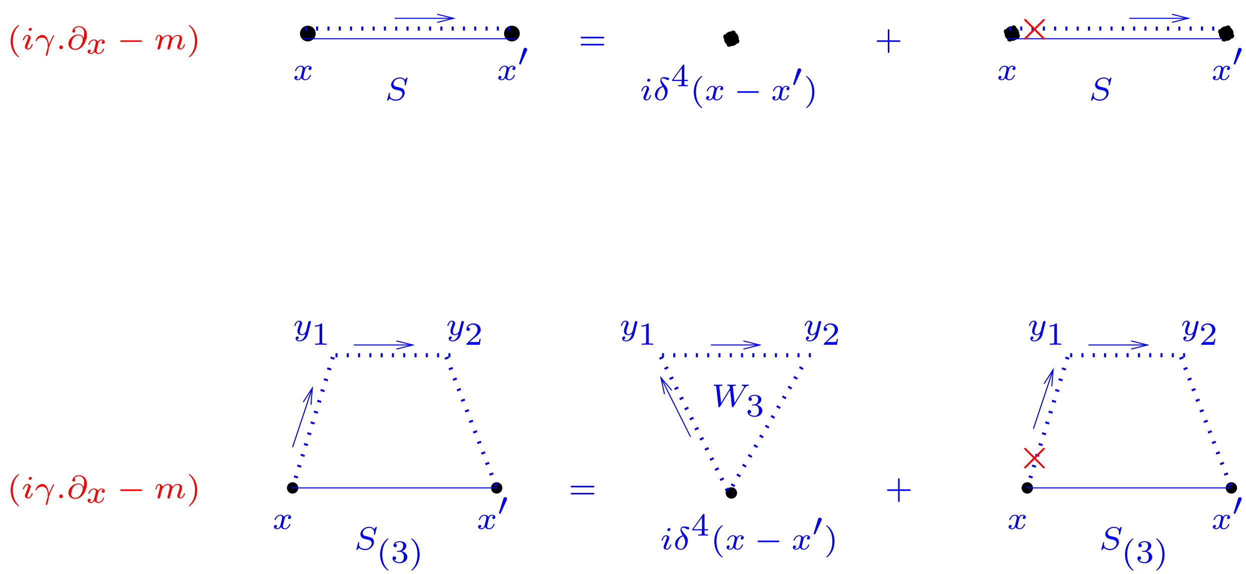
Graphical representation for $n = 3$:



Equations of motion

$$(i\gamma.\partial_{(x)} - m)S_{(n)}(x, x'; y_{n-1}, \ldots, y_1) = i\delta^4(x - x')e^{F_n(x, y_{n-1}, \ldots, y_1)}$$
$$+ i\gamma^\mu \frac{\bar{\delta} S_{(n)}(x, x'; y_{n-1}, \ldots, y_1)}{\bar{\delta} x^{\mu -}}.$$

Graphical representation of this equation for $n = 1$ and $n = 3$:



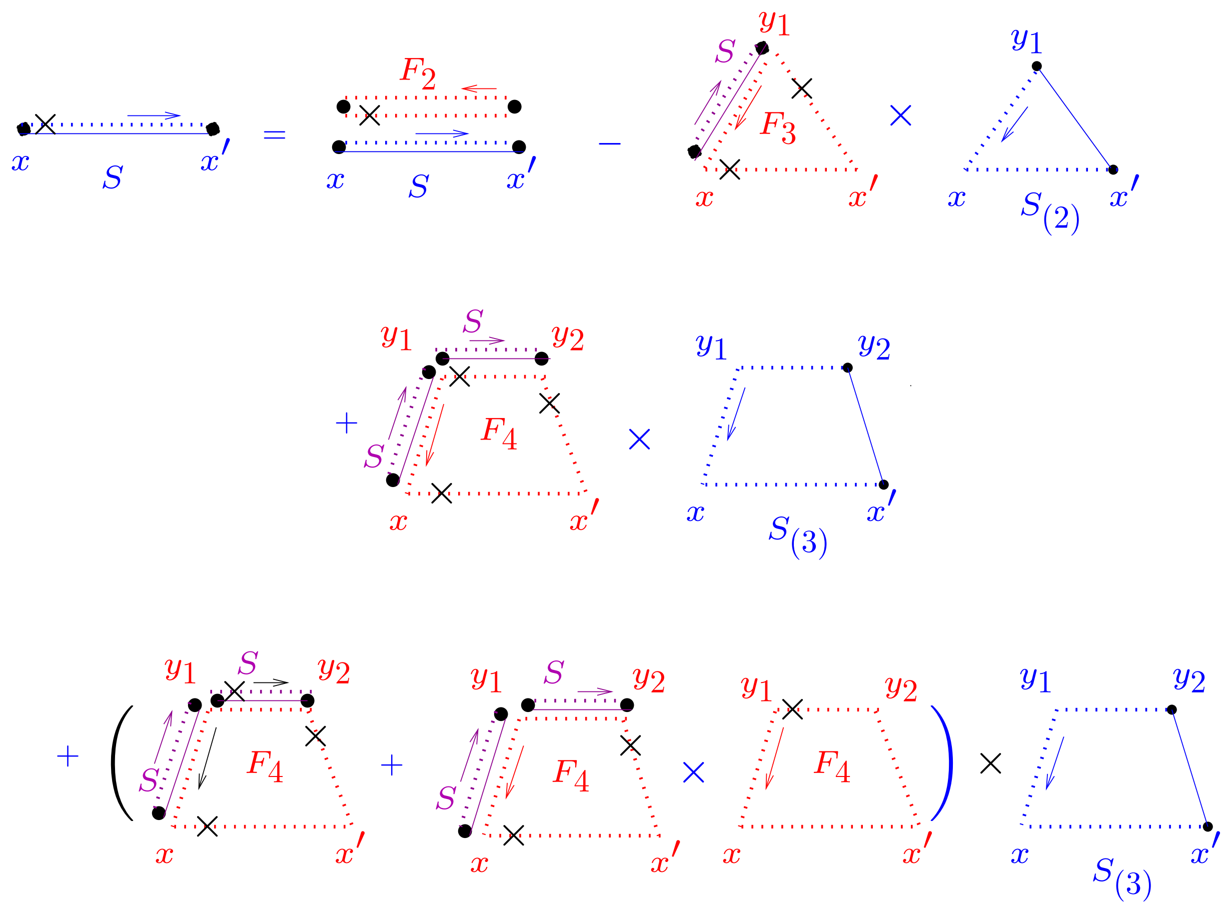
Integral equation

$\bar{\delta}S/\bar{\delta}x^{\mu-}$ and $\bar{\delta}S_{(n)}/\bar{\delta}x^{\mu-}$ can be expressed, with the aid of the functional relations, in terms of Wilson loop derivatives and Green's functions. At the end, one obtains for $\bar{\delta}S/\bar{\delta}x^{\mu-}$ a series expansion in terms of the Green's functions $S_{(n)}$, each term involving a kernel expressed in terms of Wilson loop derivatives and Green's function S and its derivative.

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} = K_{1\mu-}(x',x)\,S(x,x') + K_{2\mu-}(x',x,y_1)\,S_{(2)}(y_1,x';x)$$
$$+\sum_{n=3}^{\infty} K_{n\mu-}(x',x,y_1,\ldots,y_{n-1})\,S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2}).$$

The kernel K_n contains globally n derivatives of Wilson loops and also the Green's function S and its derivative.

Graphical representation up to third-order terms:



At short-distances, governed by perturbation theory, each derivation introduces a new power of the coupling constant and therefore the dominant terms in the expansion are the lowest-order ones. At large-distances, Wilson loops are saturated by the minimal surfaces having as supports the contours. Here also, the dominant contributions come from the lowest-order derivative terms. Therefore the expansion above can be considered in general as a perturbative one.

Thus the dominant part of the kernel comes from the second-order term (the first-order one being zero for symmetry reasons).

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu \frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}.$$

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} \simeq -\int d^4y_1 \, \frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}} \, e^{F_3(x',x,y_1)} \, S(x,y_1)\,\gamma^{\alpha_1}\,S(y_1,x').$$