Saturation in High-Energy QCD

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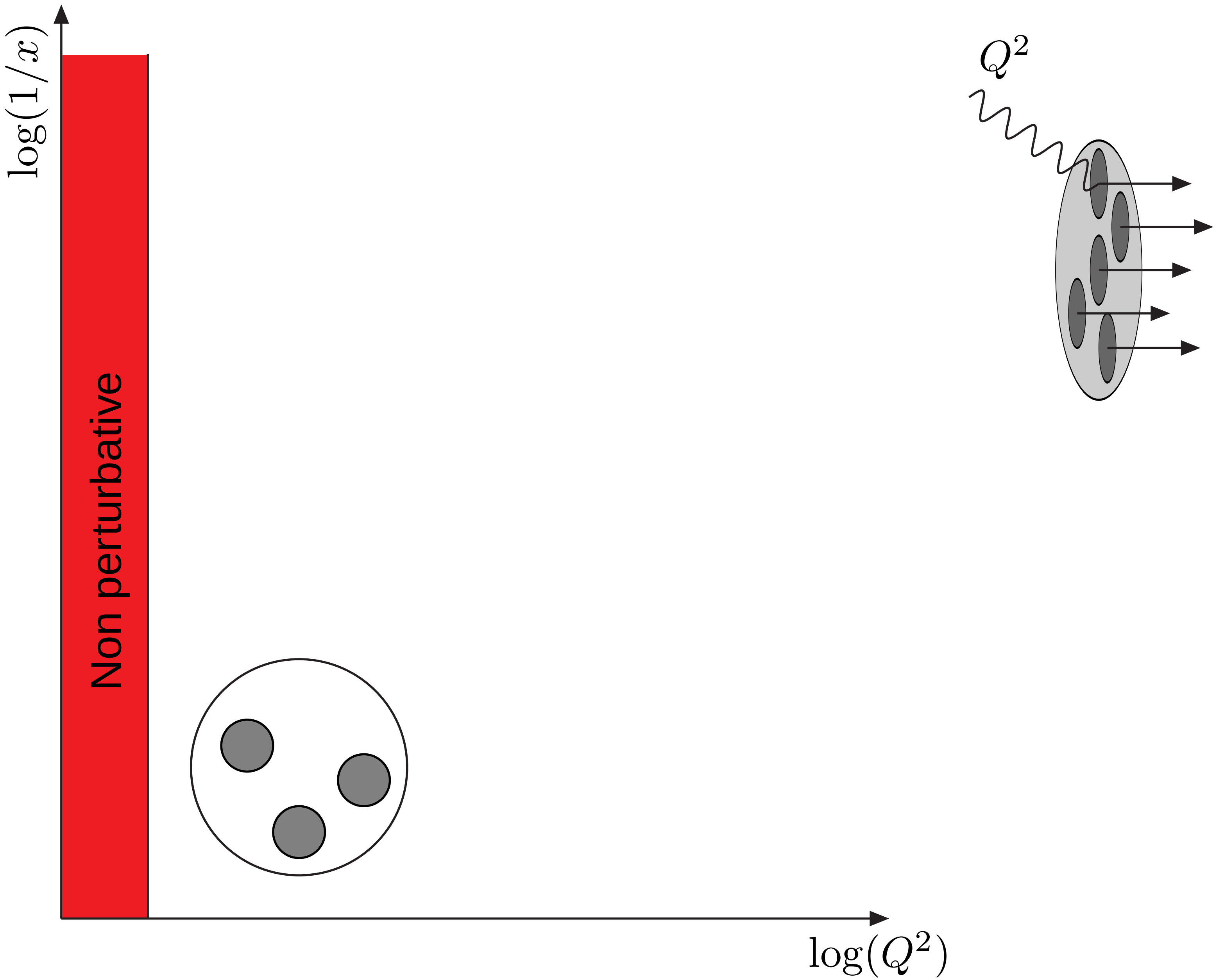
Outline

- Motivation and basic concepts: what is saturation ?
- Perturbative evolution in high-energy QCD:
  - Leading log approx.: Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation
  - Saturation effects: Balitsky-Kovchegov (BK) equation and beyond, Colour Glass Condensate
- Asymptotic solutions:
  - Equivalence with statistical physics
  - Asymptotic properties: saturation scale and geometric scaling
  - Phenomenology
- Beyond saturation: fluctuation effects
  - New equations
  - Consequences
- Outlook

Saturation effects: basics



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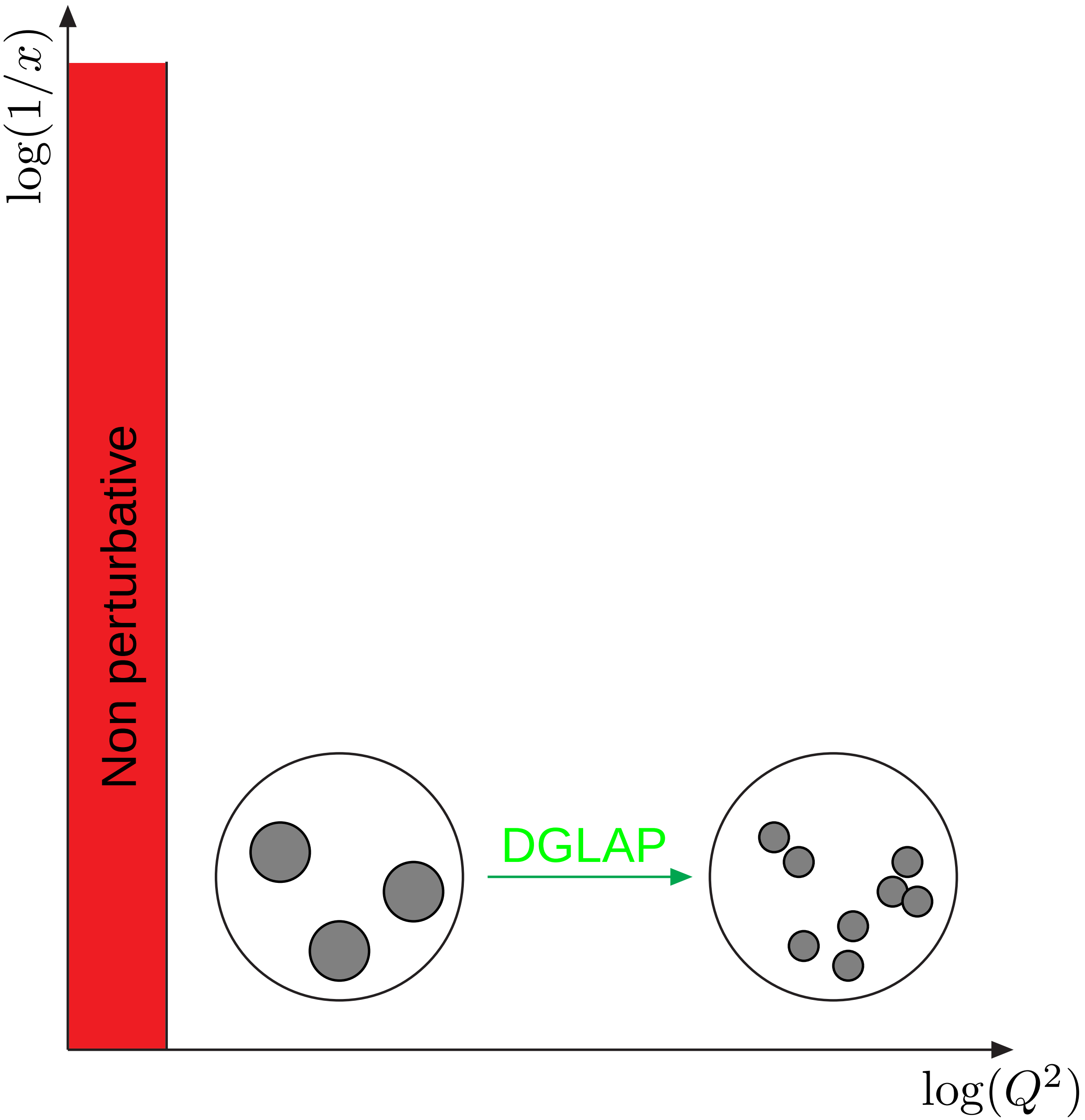

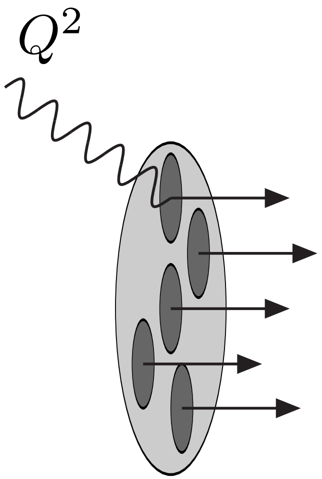

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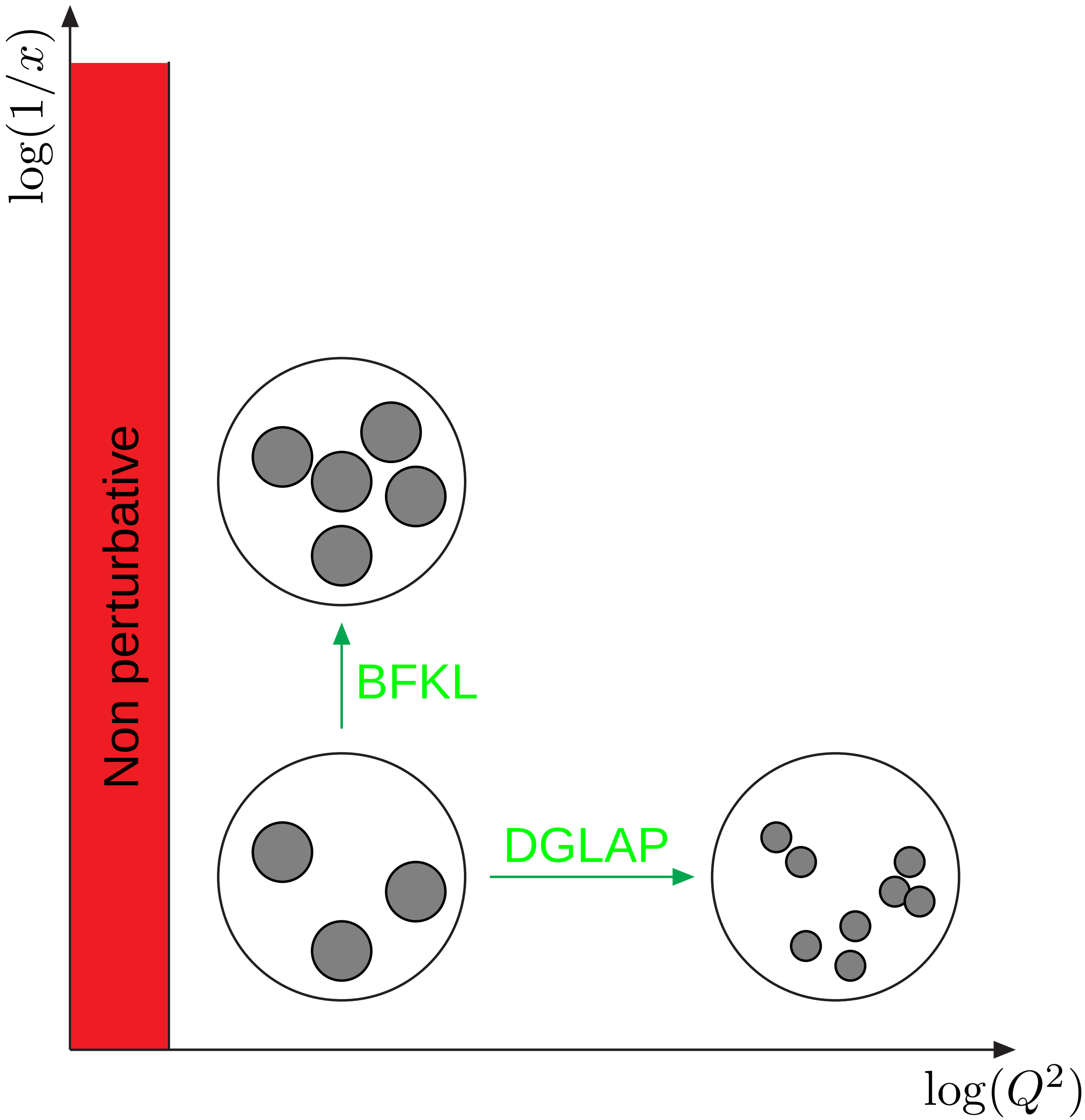

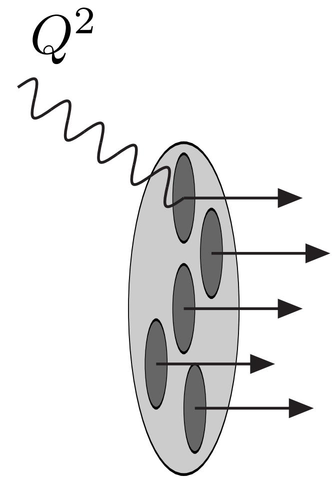

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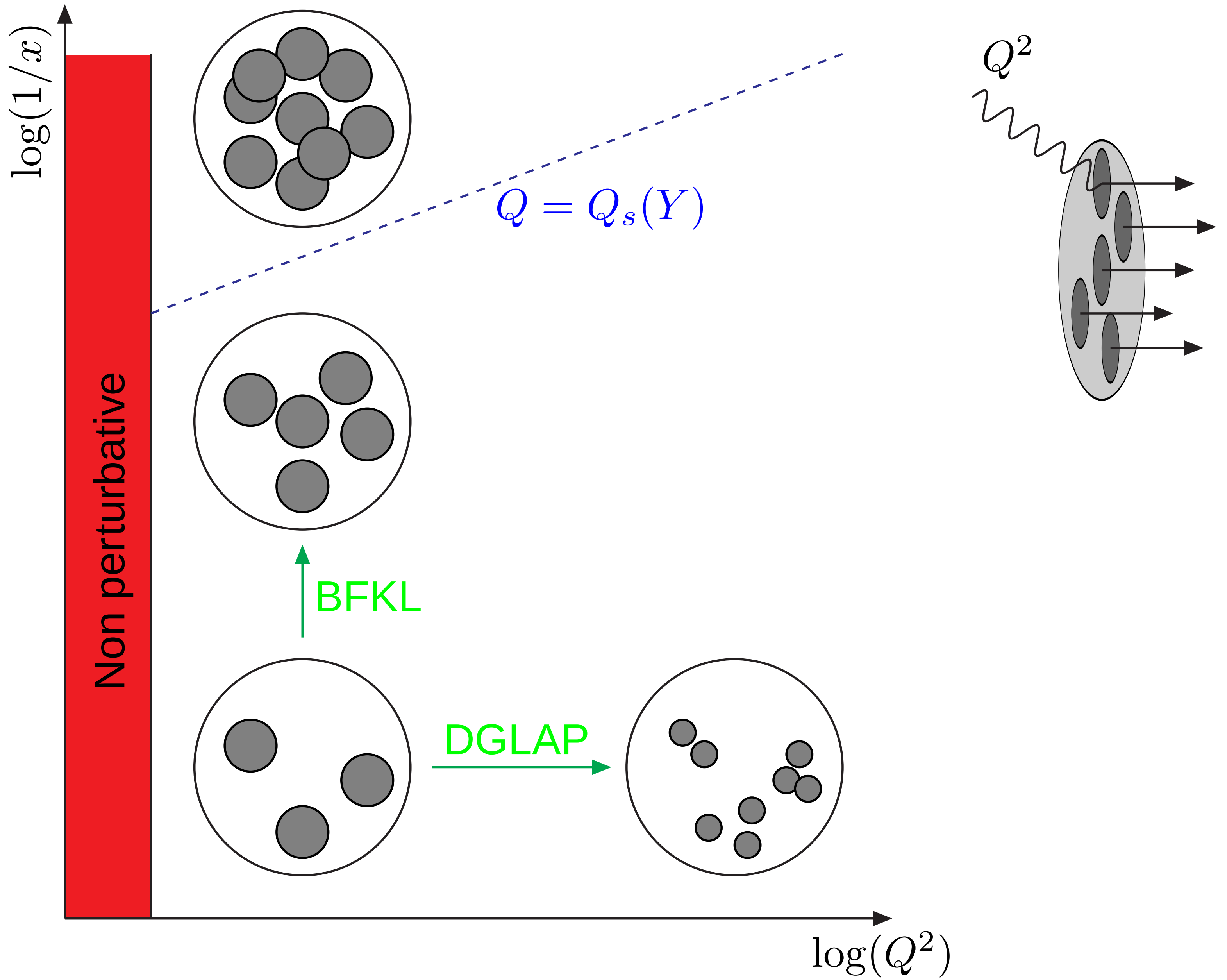

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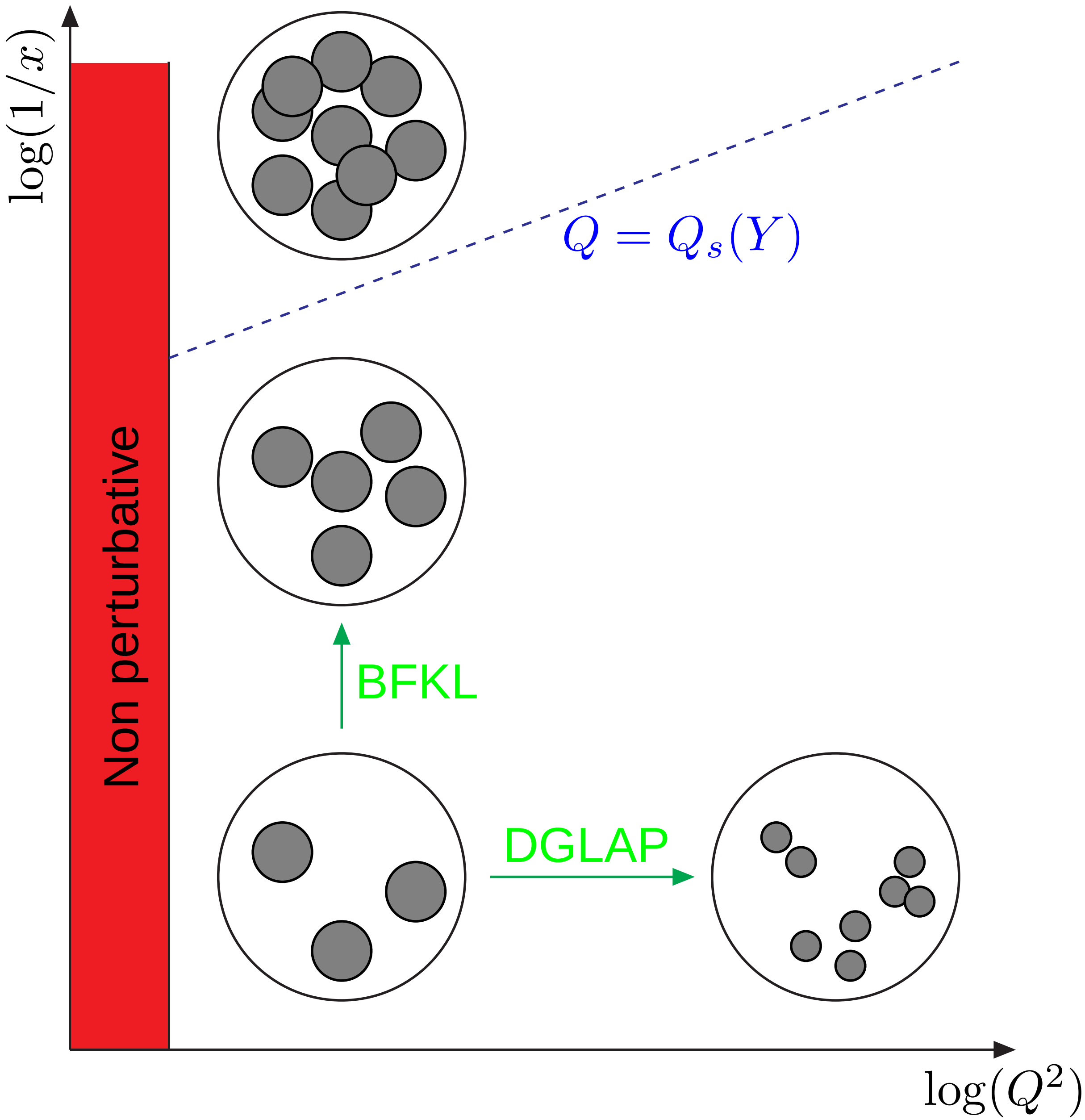
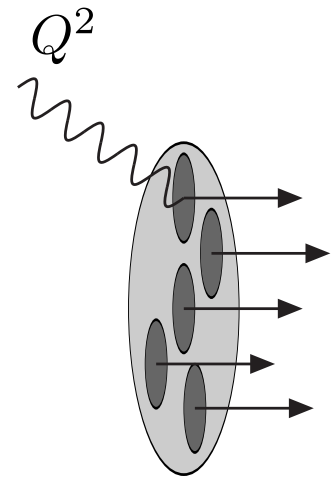
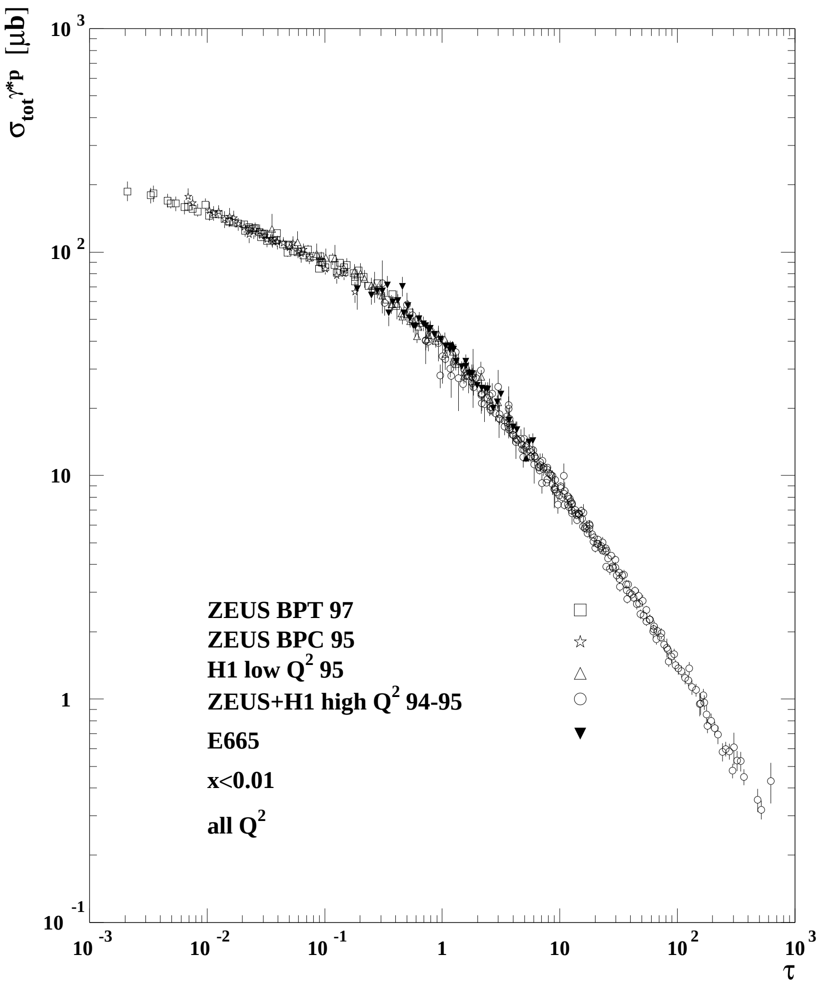




How to obtain this in QCD ?

Geometric scaling



[Stasto,Golec-Biernat,Kwiecinski,00]

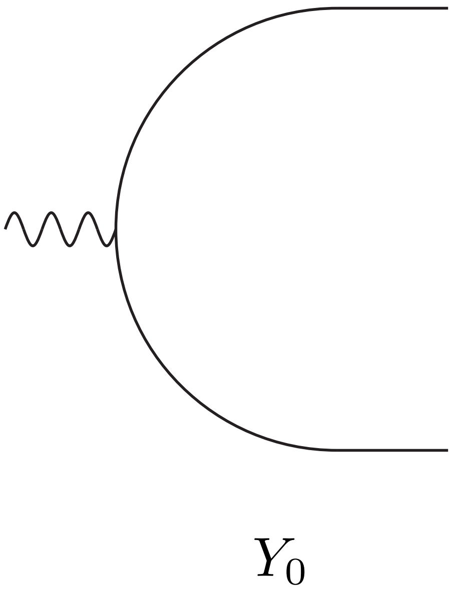
$$\sigma^{\gamma^* p}(x, Q^2) = \sigma^{\gamma^* p}\left(\frac{Q^2}{Q_s^2(Y)}\right)$$

Perturbative evolution in high-energy QCD

Dipole picture

Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

[Mueller,93]



Dipole picture

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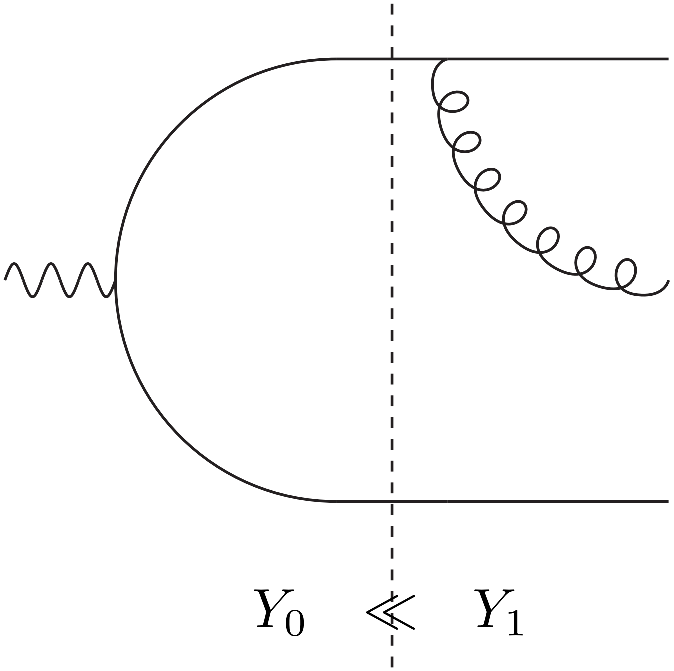

- Probability $\bar{\alpha}K$ of emission

Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

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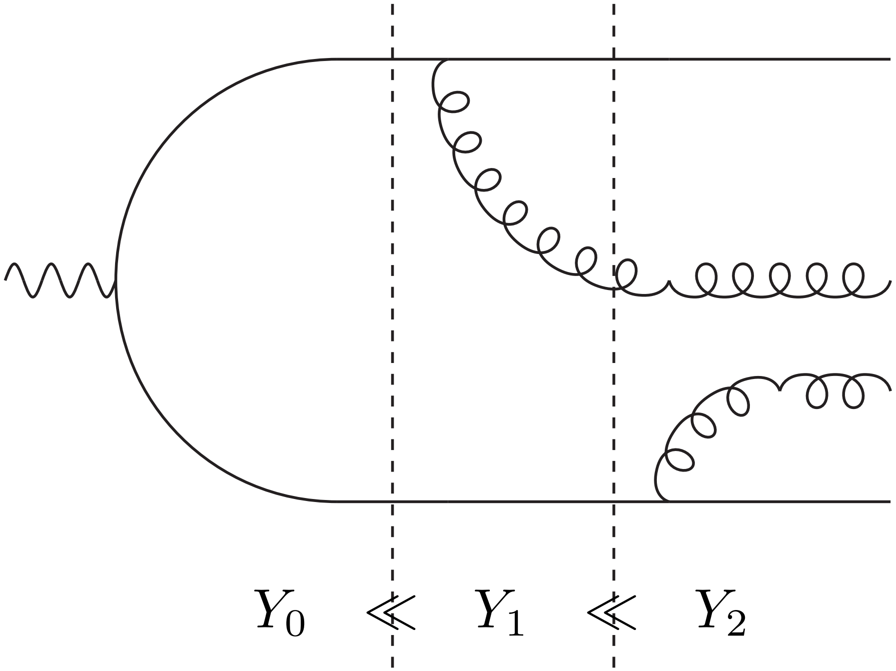

- Probability $\bar{\alpha}K$ of emission
- Independent emissions in coordinate space

Dipole picture

Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

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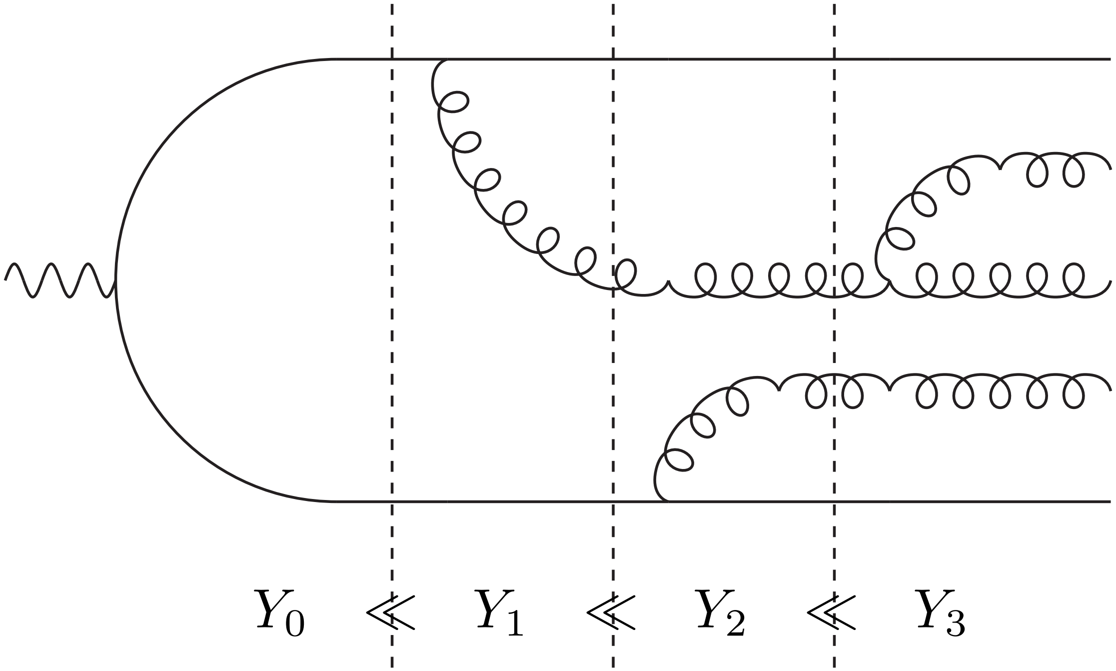

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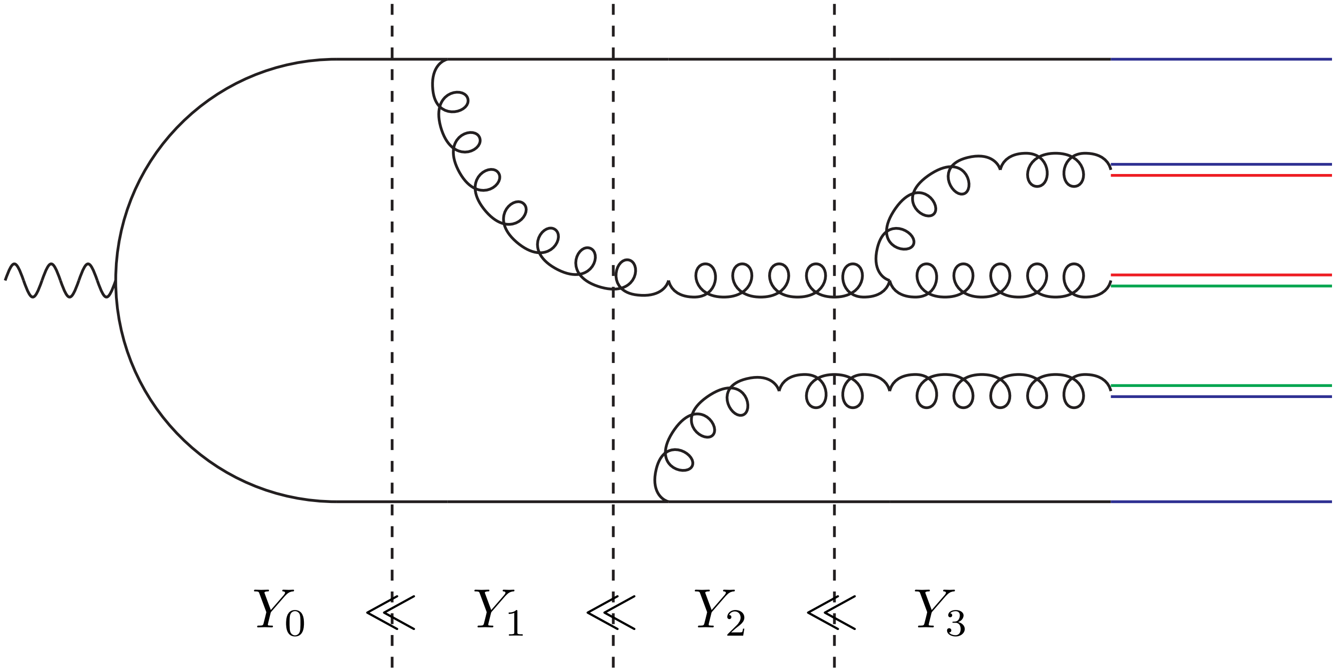
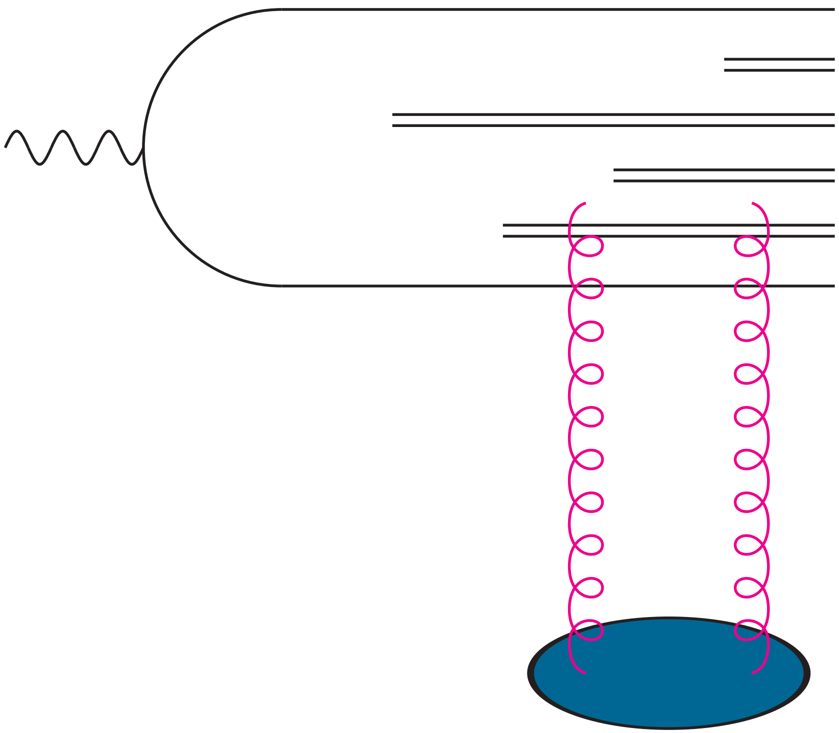


$n(r, Y)$ dipoles

of size r

- Probability $\bar{\alpha}K$ of emission
- Independent emissions in coordinate space
- Large-N_c approximation

How to observe this system ?

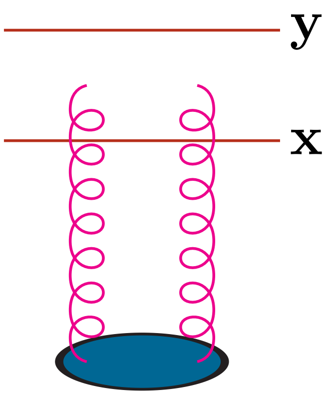


$$T(r, Y) \approx \alpha_s^2 n(r, Y)$$

Count the number of dipoles of a given size

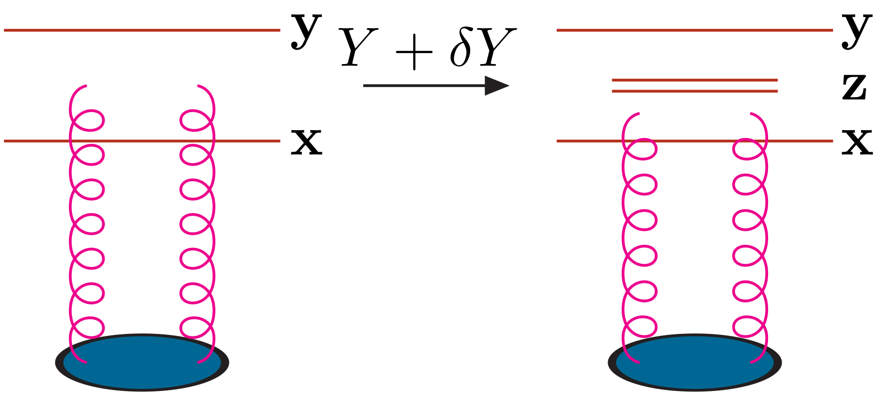
BFKL evolution (2/4)

Consider a small increase in rapidity



BFKL evolution (2/4)

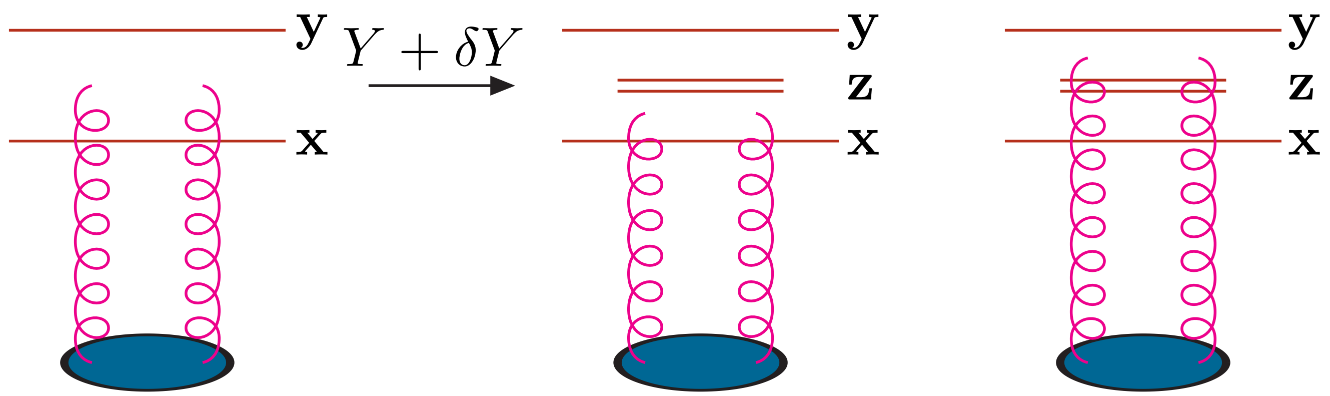
Consider a small increase in rapidity $\Rightarrow$ splitting



$\partial_Y T(\mathbf{x},\mathbf{y};Y)$

$T(\mathbf{x},\mathbf{z};Y)$

Consider a small increase in rapidity $\Rightarrow$ splitting

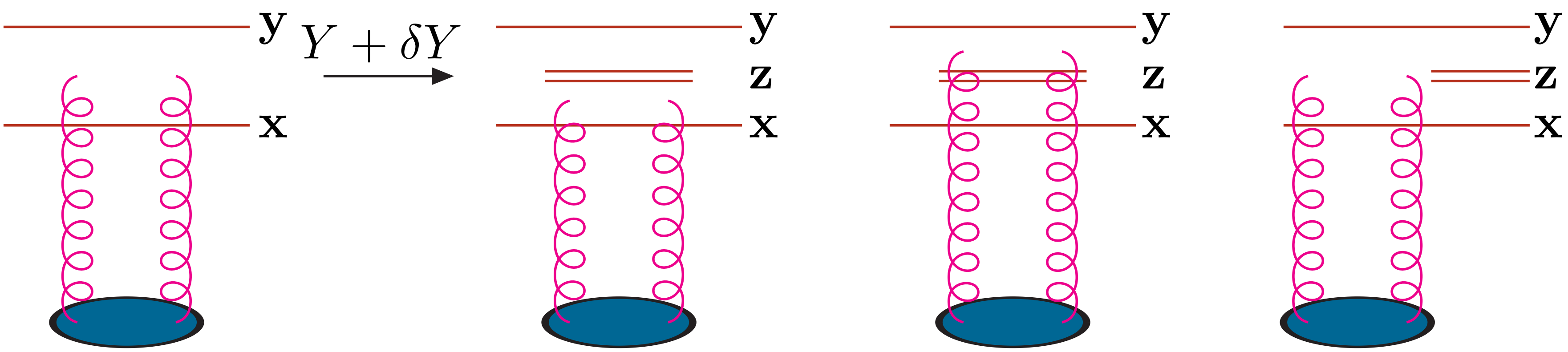


$$\partial_Y T(\mathbf{x},\mathbf{y};Y)$$

$$T(\mathbf{x},\mathbf{z};Y) + T(\mathbf{z},\mathbf{y};Y)$$

BFKL evolution (2/4)

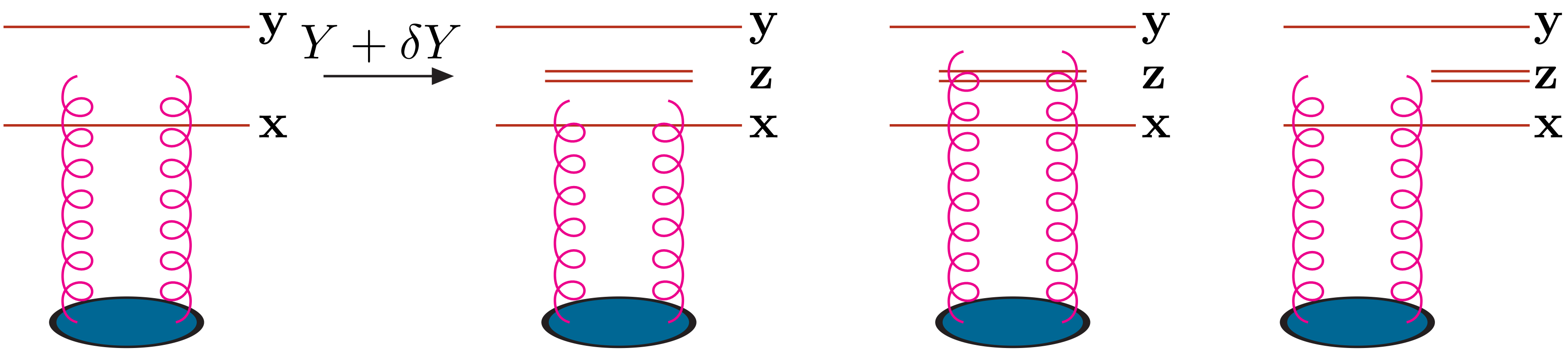
Consider a small increase in rapidity $\Rightarrow$ splitting



$$\partial_Y T(\mathbf{x}, \mathbf{y}; Y)$$

$$T(\mathbf{x}, \mathbf{z}; Y) + T(\mathbf{z}, \mathbf{y}; Y) - T(\mathbf{x}, \mathbf{y}; Y)$$

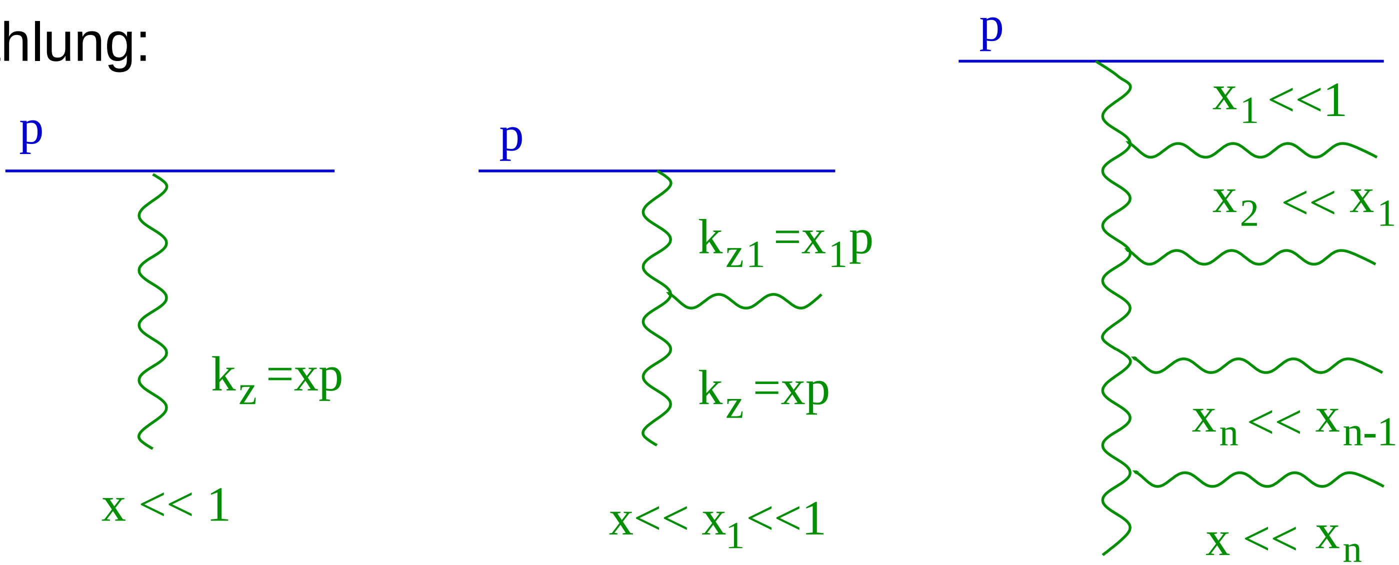
Consider a small increase in rapidity $\Rightarrow$ splitting



$$\partial_Y T(\mathbf{x},\mathbf{y};Y) = \bar{\alpha}\int d^2z\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[T(\mathbf{x},\mathbf{z};Y)+T(\mathbf{z},\mathbf{y};Y)-T(\mathbf{x},\mathbf{y};Y)\right]$$

[Balitsky,Fadin,Kuraev,Lipatov,78]

Bremsstrahlung:



Probability of emission

$$dP \sim \alpha_s \frac{dk^2}{k^2} \frac{dx}{x}$$

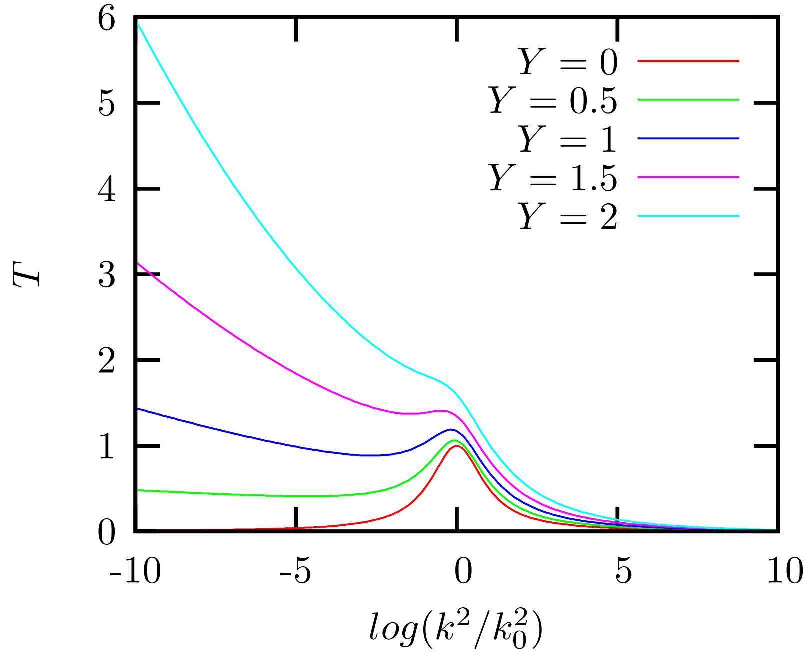
In the small-x limit

$$\int_x^1 \frac{dx_1}{x_1} \sim \alpha_s \log(1/x)$$

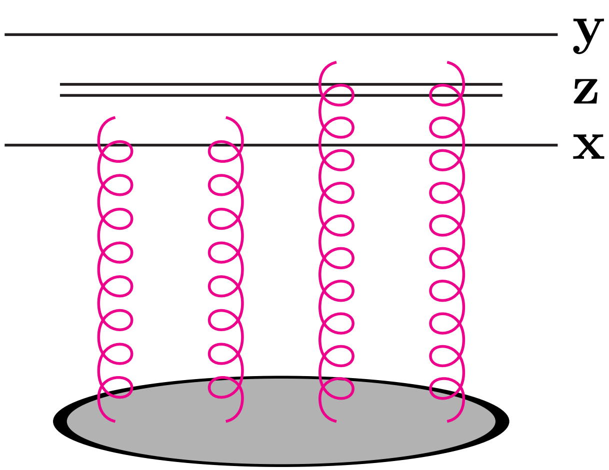
The solution goes like

$$T(Y) \sim e^{\omega Y} \qquad \text{with} \quad \omega = 4\bar{\alpha}\log(2) \approx 0.5$$

- Fast growth of the amplitude
- Intercept value too large
- Violation of the Froissart bound:
  $T(Y) \le C \log^2(s) \qquad T(r,b) \le 1$



+ problem of diffusion in the infrared

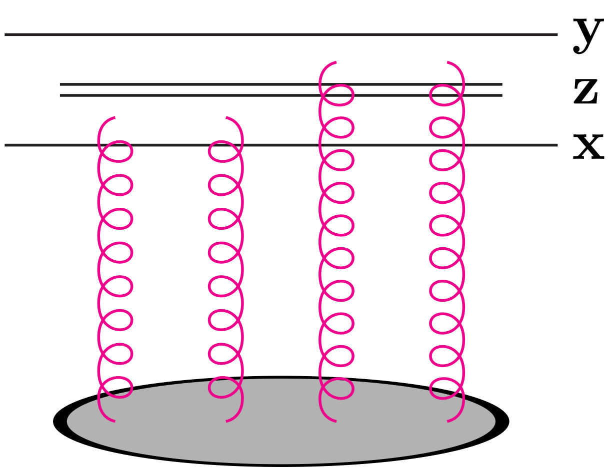


Multiple scattering

- ⋆ Proportional to T^2
- ⋆ important when $T \approx 1$

$$\partial_Y T(\mathbf{x},\mathbf{y};Y) = \bar{\alpha}\int d^2z\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[T(\mathbf{x},\mathbf{z};Y)+T(\mathbf{z},\mathbf{y};Y)-T(\mathbf{x},\mathbf{y};Y)\right.$$
$$\left.-T(\mathbf{x},\mathbf{z};Y)T(\mathbf{z},\mathbf{y};Y)\right]$$

Saturation effects



Multiple scattering

- $\star$ Proportional to T^2
- $\star$ important when $T \approx 1$

$$\partial_Y \langle T(\mathbf{x},\mathbf{y};Y)\rangle$$

$$= \bar{\alpha} \int d^2z \, \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left[\langle T(\mathbf{x},\mathbf{z};Y)\rangle + \langle T(\mathbf{z},\mathbf{y};Y)\rangle - \langle T(\mathbf{x},\mathbf{y};Y)\rangle \right.$$

$$\left. - \langle T(\mathbf{x},\mathbf{z};Y) T(\mathbf{z},\mathbf{y};Y)\rangle \right]$$

contains

$$\partial \langle T(\mathbf{x},\mathbf{y};Y)\rangle \longrightarrow \langle T(\mathbf{x},\mathbf{z};Y) T(\mathbf{z},\mathbf{y};Y)\rangle$$

Balitsky vs. Kovchegov

In general: complete hierarchy [Balitsky, 96]

$$\partial_Y \langle T^k \rangle \longrightarrow \underbrace{\langle T^k \rangle}_{\text{BFKL}}, \underbrace{\langle T^{k+1} \rangle}_{\text{saturation}}$$

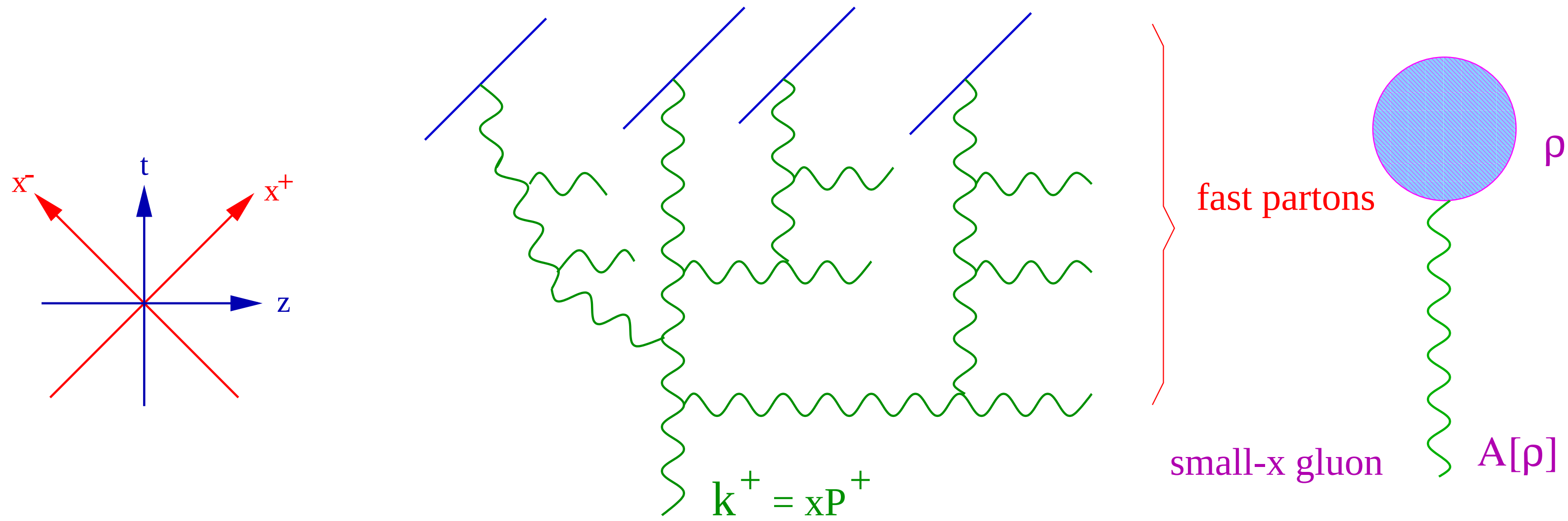
Mean field approx.: $\langle T_{\mathbf{xz}} T_{\mathbf{zy}} \rangle = \langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle$

$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \, \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2 (\mathbf{z}-\mathbf{y})^2} \left[\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle - \langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle \right]$$

[Balitsky 96,Kovchegov 99]

Simplest perturbative evolution equation satisfying unitarity constraint

Color Glass Condensate (1/4)

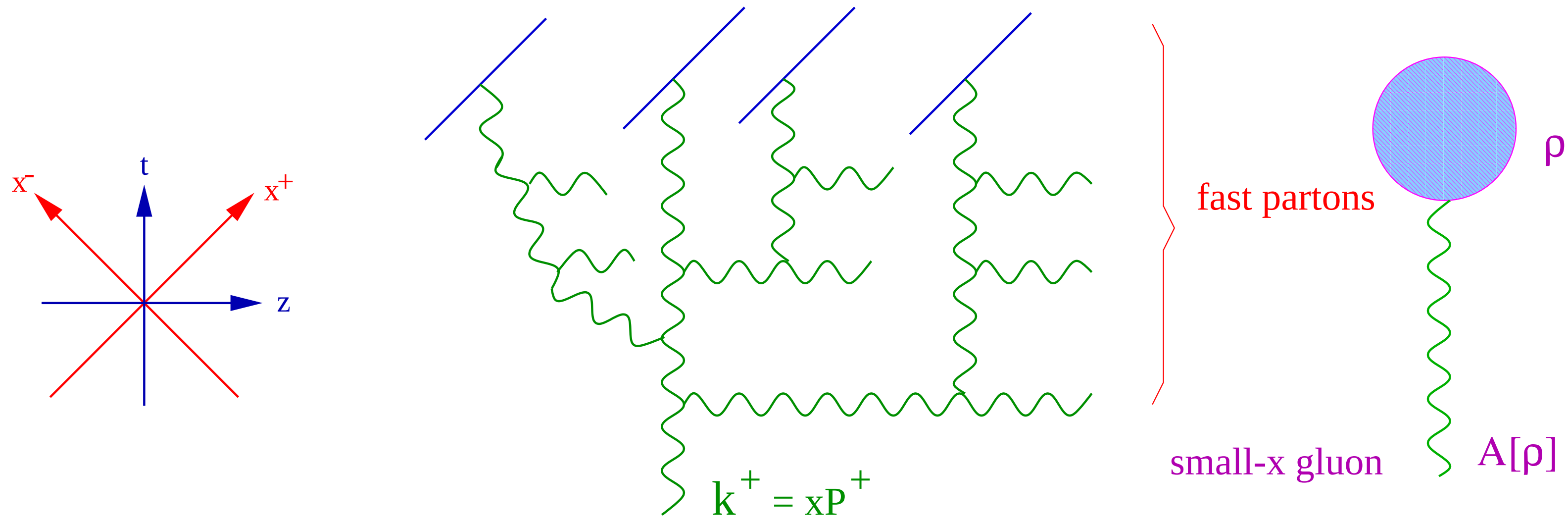


Effective theory for High-Energy QCD:

- Theory for the **gluonic field**: Color
- Small-x $\equiv$ classical field radiated by frozen fast gluons
  Large-x $\equiv$ random distribution of color sources: Glass
- Large occupation number: Condensate

Equation for the probability distribution of the color charge $W_Y[\rho]$

Color Glass Condensate (2/4)



- fast gluons: frozen, source for slow partons

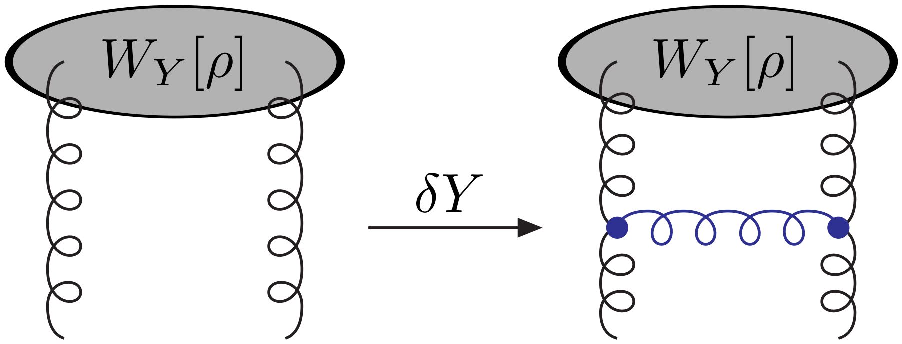
$$(D_\mu F^{\mu\nu})_a = \delta^{\nu +}\rho_a(x^-, \mathbf{x}_\perp)$$

- Random source: correlators computed using the probability distribution $W_Y[\rho]$

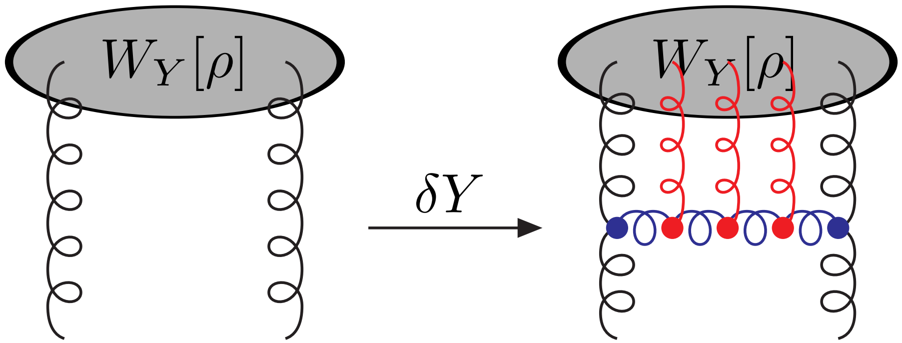
$$\langle A^i_a A^i_a\rangle = \int \mathcal{D}\rho\, W_Y[\rho]\, A^i_a A^i_a$$

- Strong field $A \sim 1/g$ (equivalent to $n \sim 1/\alpha_s$ or $T \sim 1$)

Color Glass Condensate (3/4)



Color Glass Condensate (3/4)



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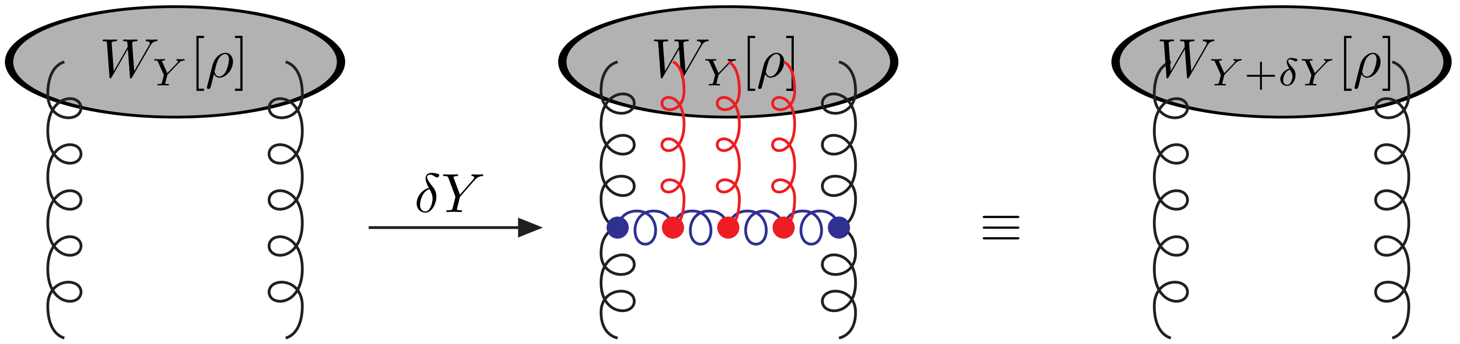


Evolution

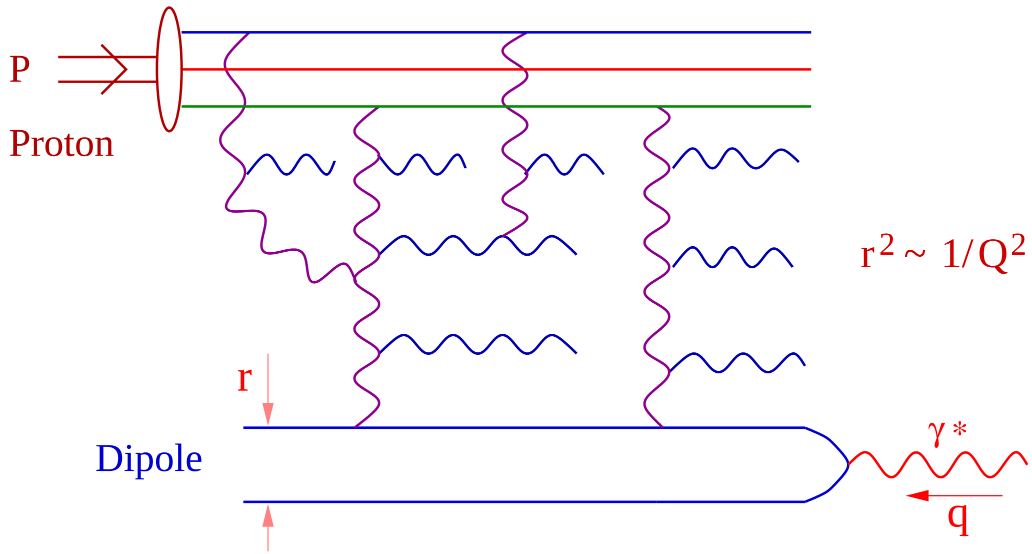
$$\partial_Y W_Y[\rho] = \frac{1}{2}\int_{\mathbf{xy}} \frac{\delta}{\delta\rho^a_{\mathbf{x}}}\chi^{ab}_{\mathbf{xy}}[\rho]\frac{\delta}{\delta\rho^a_{\mathbf{y}}}W_Y[\rho]$$

[Jalilan-Marian,Iancu,McLerran,Weigert,Leonidov,Kovner,97-00]

Wilson Line

$$V_x = P\,\exp\left[ig\int dx^-\,A^+(x^-,\mathbf{x}_\perp)\right]$$

In the weak field limit $\longrightarrow$ BFKL.



S-matrix: $\gamma^* \to q\bar{q} \to V_{\mathbf{x}}^\dagger V_{\mathbf{y}}$

$$S_Y = \int \mathcal{D}A^+ \, W_Y[A] \, \frac{1}{N_c} \mathrm{tr}(V_{\mathbf{x}}^\dagger V_{\mathbf{y}})$$

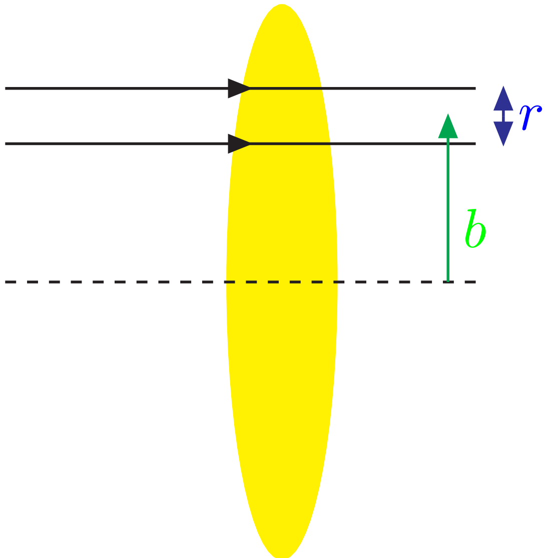
Wilson Line

$$V_x = P \, \exp\left[ig \int dx^- \, A^+(x^-, \mathbf{x}_\perp)\right]$$

Solutions

The BK equation

The b independent case



$$T(\mathbf{x},\mathbf{y})$$

$$\downarrow$$

$$T(\mathbf{r};\mathbf{b})$$

$$\downarrow$$

$$T(\mathbf{r})$$

Momentum space:

$$T(\mathbf{k}) = \frac{1}{2\pi}\int \frac{d^2r}{r^2} e^{i\mathbf{r}.\mathbf{k}} T(\mathbf{r}) = \int \frac{dr^2}{r^2} J_0(kr) T(r)$$

Link with statistical physics (1/2)

BK equation:

$$\partial_Y T(k) = \underbrace{\frac{\bar\alpha}{\pi}\int \frac{dp^2}{p^2}\left[\frac{p^2T(p)-k^2T(k)}{|k^2-p^2|}+\frac{k^2T(k)}{\sqrt{4p^4+k^4}}\right]}_{\bar\alpha\chi(-\partial_L)} - \bar\alpha T^2(k)$$

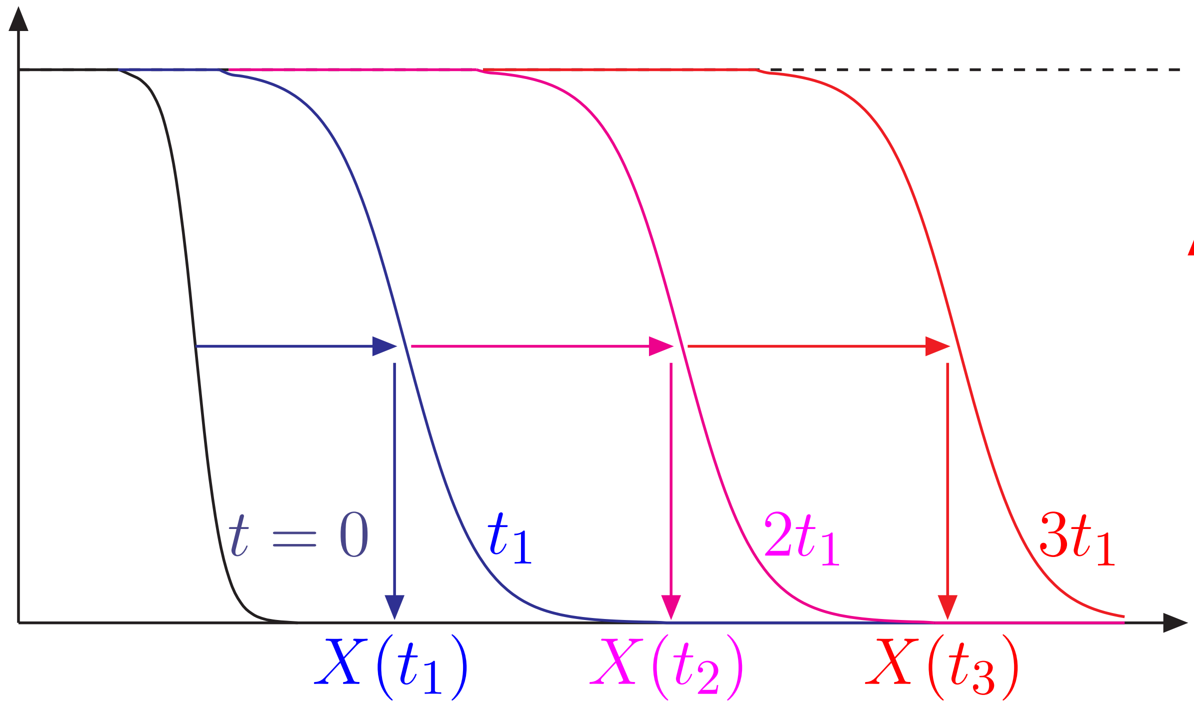
Saddle point/diffusive approximation:

$$\chi(\gamma) = \chi\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\chi''\left(\tfrac{1}{2}\right)\left(\gamma-\tfrac{1}{2}\right)^2$$

+ linear change of variable: $t=\bar\alpha Y$, $x \approx L = \log(k^2/k_0^2)$

$\longrightarrow$ Fisher-Kolmogorov-Petrovsky-Piscounov equation

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u^2(x,t)$$

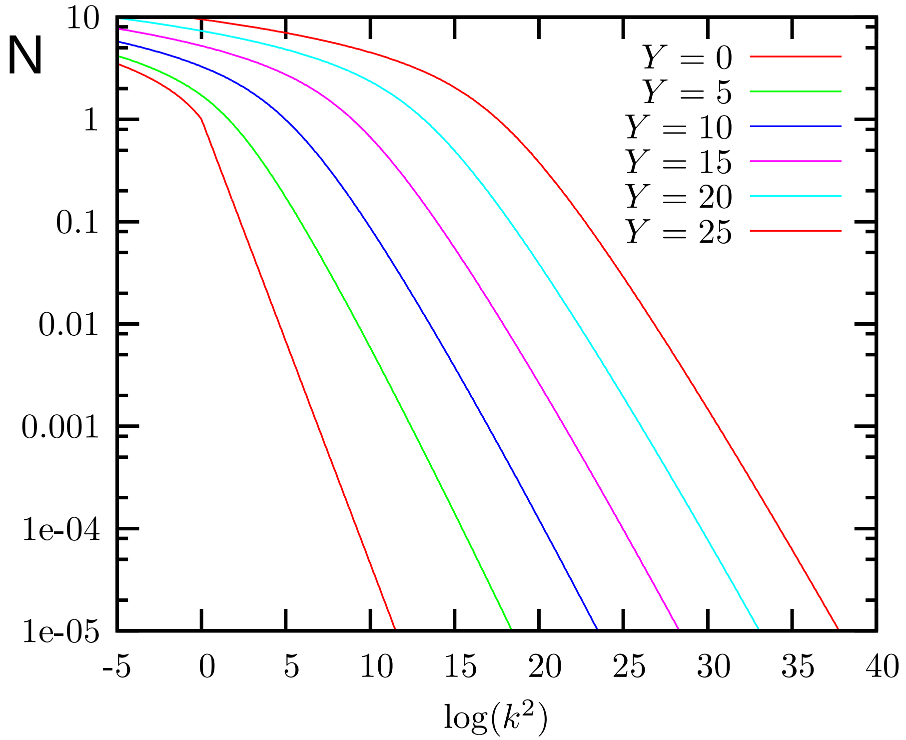


Asymptotic solution:
traveling wave

$$u(x,t) = u(x - v_c t)$$

Position: $X(t) = X_0 + v_c t$

Numerical simulations: $\bar{\alpha} = 0.2$



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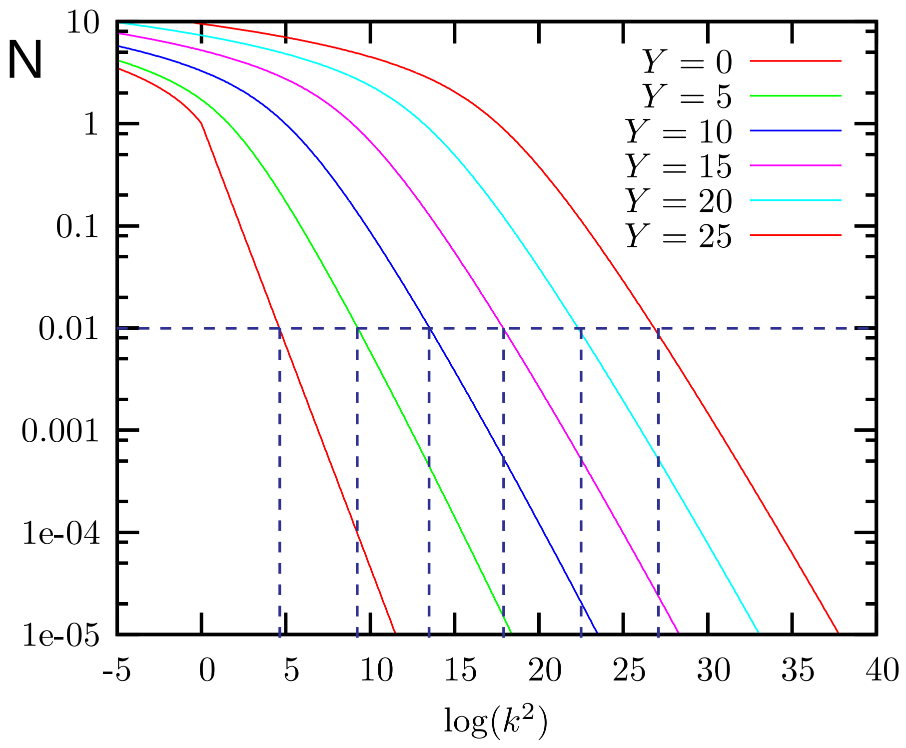
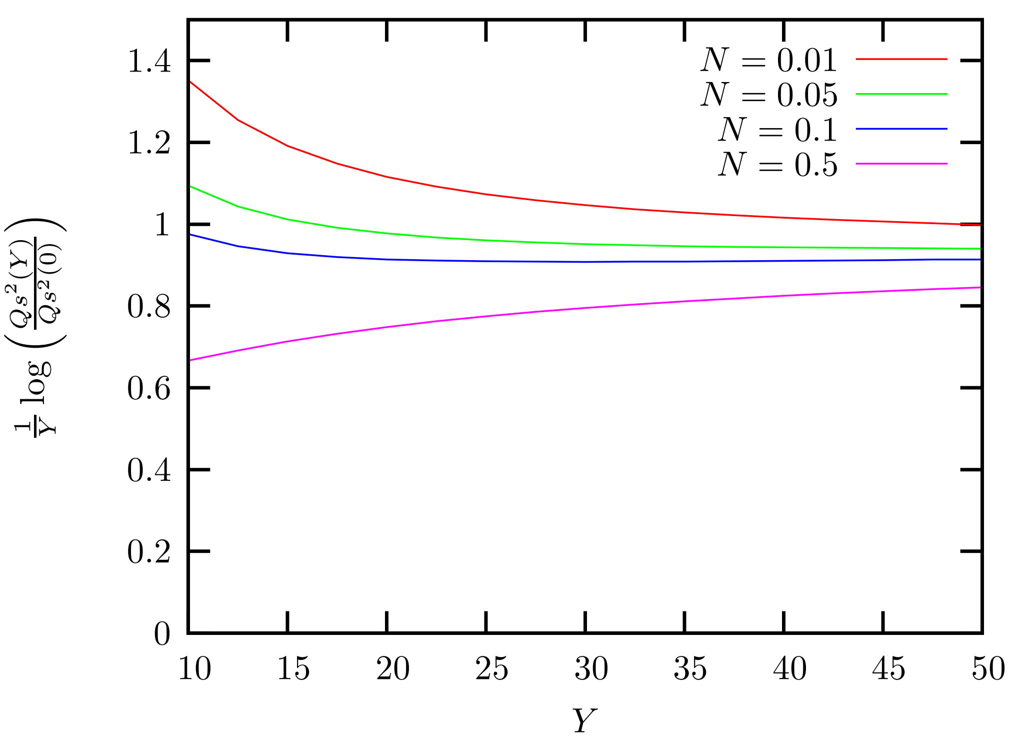




Geometric scaling

$\Rightarrow$ *geometric scaling from the BK equation*

Saturation Scale

$$Q_s^2(Y) \sim k_0^2 \exp\left[v_c Y - \frac{3}{2\gamma_c}\log(Y) - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}}\right]$$

Tail of the front

$$\begin{aligned} T(k,Y) &= T\left(\frac{k^2}{Q_s^2(Y)}\right) \\ &\approx \log\left(\frac{k^2}{Q_s^2(Y)}\right)\left|\frac{k^2}{Q_s^2(Y)}\right|^{-\gamma_c} \end{aligned}$$

The full BK equation

Can we extend this including the b dependence

Go to momentum space

$$\tilde{T}(\mathbf{k},\mathbf{q}) = \int d^2x\, d^2y\, e^{i\mathbf{k}.\mathbf{x}} e^{i(\mathbf{q}-\mathbf{k}).\mathbf{y}} \frac{T(\mathbf{x},\mathbf{y})}{(\mathbf{x}-\mathbf{y})^2}$$

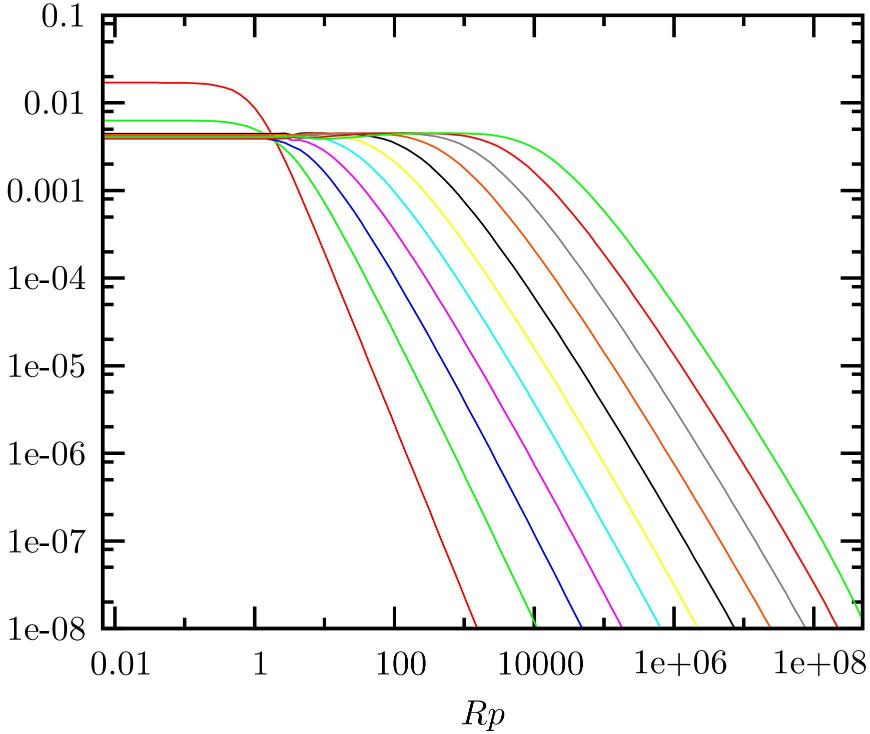
new form of the BK equation

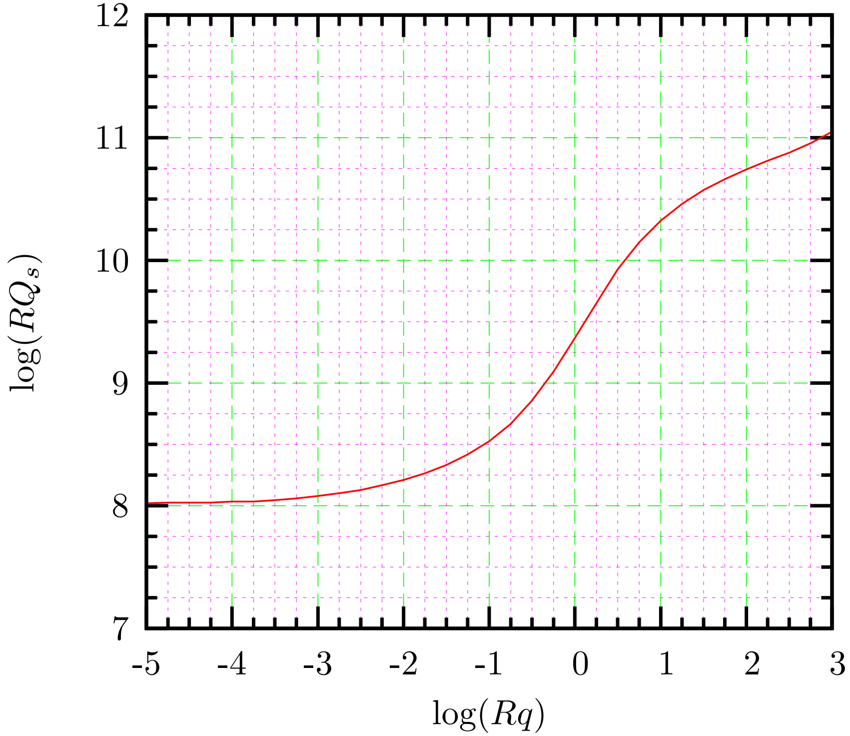
$$\begin{aligned}\partial_Y \tilde{T}(\mathbf{k},\mathbf{q}) &= \frac{\bar{\alpha}}{\pi} \int \frac{d^2k'}{(k-k')^2} \left\{ \tilde{T}(\mathbf{k}',\mathbf{q}) - \frac{1}{4}\left[\frac{k^2}{k'^2} + \frac{(q-k)^2}{(q-k')^2}\right] \tilde{T}(\mathbf{k},\mathbf{q}) \right\} \\ &- \frac{\bar{\alpha}}{2\pi} \int d^2k'\, \tilde{T}(\mathbf{k},\mathbf{k}')\tilde{T}(\mathbf{k}-\mathbf{k}',\mathbf{q}-\mathbf{k}')\end{aligned}$$

[C.Marquet, R.Peschanski, G.S., 05]

Numerical simulations

Dependence on momentum transfer k: traveling waves





One can prove analytically that:

- formation of a traveling wave at large p (or k)
- q dependence: scales like a constant or linearly ($Y = 25$)

Phenomenology (1/2)

BK evolution ($L = \log(k^2)$) [C.Marquet, R.Peschanski, G.S., 05]

$$\partial_Y N(L,Y) = \underbrace{\chi(-\partial_L)}_{A_0 - A_1\partial_L + A_2\partial_L^2} N(L,Y) - N^2(L,Y)$$

Search explicit parametric traveling-waves solutions $N(L,Y) \equiv A_0 U(s)$ with $s \equiv \frac{\lambda}{c}\, L - \left(A_0 + \frac{\lambda}{c} A_1\right) Y$, ($\lambda = \sqrt{A_0/A_2}$)

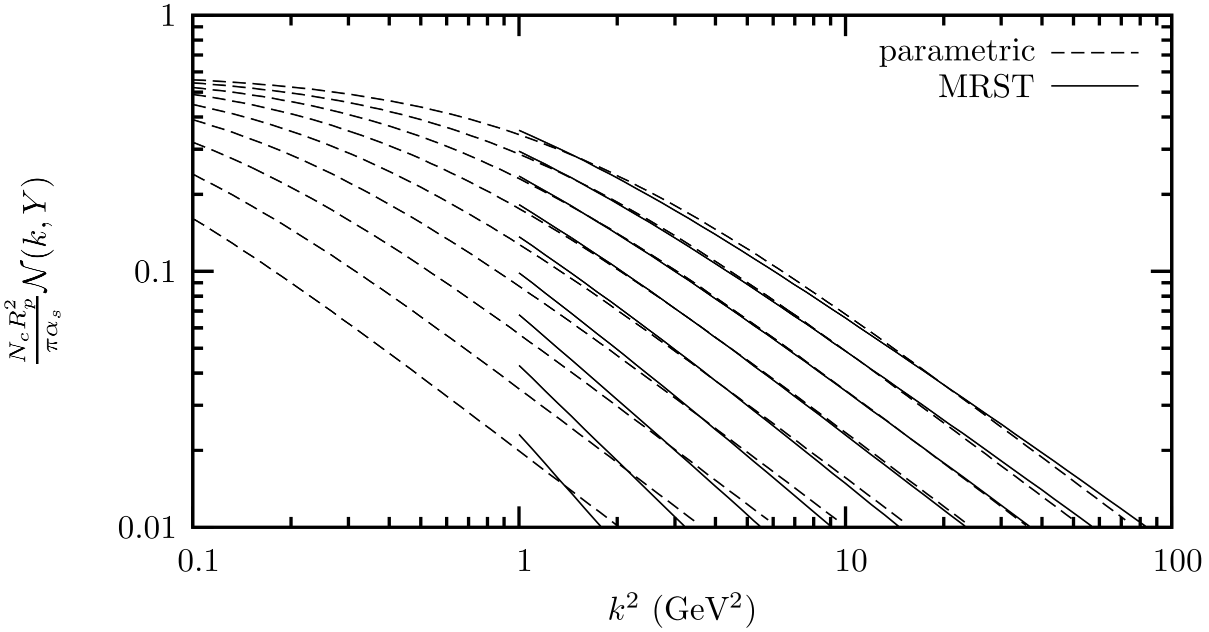
$$\frac{1}{c^2} U''(s) + U'(s) + U(s) - U^2(s) = 0$$

$1/c$ treated as a small parameter $\longrightarrow$ approximate solutions

$$U(s) = \frac{1}{1+e^s} - \frac{1}{c^2}\, \frac{e^s}{(1+e^s)^2}\, \log\left[\frac{(1+e^s)^2}{4e^s}\right] - \frac{\lambda^3}{c^3}\, \frac{A_3}{A_0}\, \frac{e^s}{(1+e^s)^2}\, \left[3\frac{(1-e^s)}{(1+e^s)} + s\right] + \mathcal{O}\left(\frac{1}{c^4}\right)$$

Note: can be extended for running coupling

Compare with the MRST gluon distribution

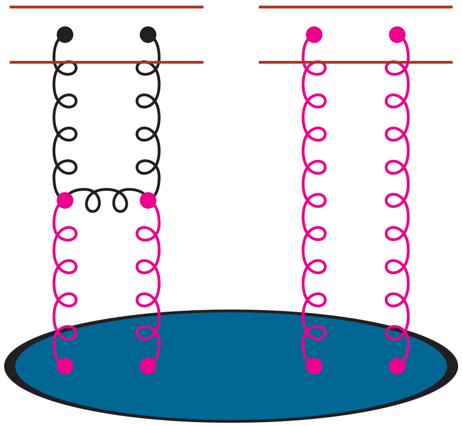


- Traveling-wave pattern
- Interior of the wave *i.e.* approach to saturation
- Works at non-asymptotic energies: $Y \leq 12$
- Adjust kernel parameters: NLO effects ?

Fluctuations

[E. Iancu, D. Triantafyllopoulos]
Also A. Mueller, S. Munier, A. Shoshi,
W. van Saarloos, S. Wong

Consider correlations $\langle T^{(k)} \rangle$



- Usual BFKL ladder

$T^{(k)} \to T^{(k)}$

[E. Iancu, D. Triantafyllopoulos]
Also A. Mueller, S. Munier, A. Shoshi,
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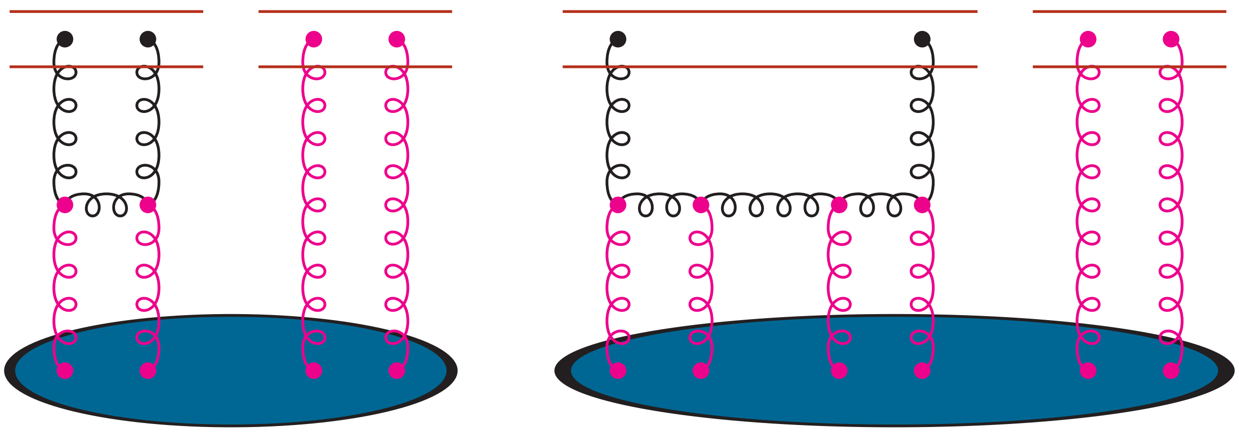
Consider correlations $\langle T^{(k)} \rangle$

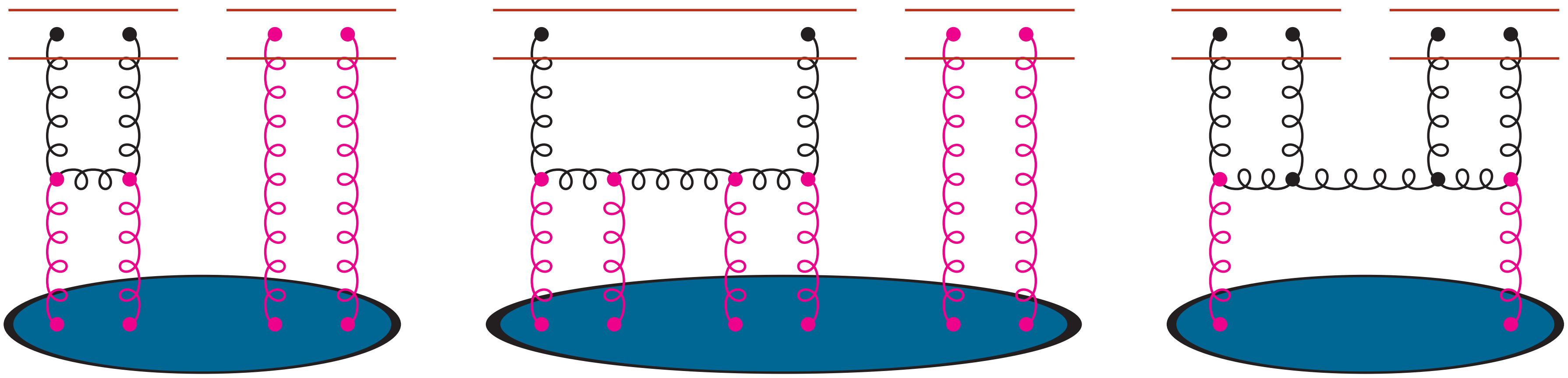


- Usual BFKL ladder $T^{(k)} \rightarrow T^{(k)}$
- fan diagram $\longrightarrow$ saturation effects $T^{(k+1)} \rightarrow T^{(k)}$

Fluctuations

Consider correlations $\langle T^{(k)} \rangle$

[E. Iancu, D. Triantafyllopoulos]
Also A. Mueller, S. Munier, A. Shoshi, W. van Saarloos, S. Wong



- Usual BFKL ladder $T^{(k)} \to T^{(k)}$
- fan diagram $\longrightarrow$ saturation effects $T^{(k+1)} \to T^{(k)}$
- splitting $\longrightarrow$ fluctuations, pomeron loops $T^{(k-1)} \to T^{(k)}$

Evolution hierarchy

$\Rightarrow$ complicated hierarchy

$$\begin{aligned}
&\partial_Y T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{y}_2;Y)\\
&= \quad \bar\alpha \int d^2z\, \frac{(\mathbf{x}_1-\mathbf{y}_1)^2}{(\mathbf{x}_1-\mathbf{z})^2(\mathbf{z}-\mathbf{y}_1)^2}\left[T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{z};Y)+T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{z},\mathbf{y}_2;Y)\right.\\
&\qquad\left. -T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{y}_2;Y)-T^{(3)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{z};\mathbf{z},\mathbf{y}_2;Y)+(1\leftrightarrow 2)\right]\\
&+ \quad \bar\alpha\alpha_s^2\kappa\,\frac{(\mathbf{x}_1-\mathbf{y}_1)^2(\mathbf{x}_2-\mathbf{y}_2)^2}{(\mathbf{x}_1-\mathbf{y}_2)^2}\,T^{(1)}(\mathbf{x}_1,\mathbf{y}_2;Y)\,\delta^{(2)}(\mathbf{y}_1-\mathbf{x}_2).
\end{aligned}$$

- Merging term: important when $T^{(2)}\sim T^{(1)}\sim 1$ i.e. at saturation
- Splitting term: important when $T^{(2)}\sim\alpha_s^2 T^{(1)}$ or $T\sim\alpha_s^2$ i.e. in the dilute regime

Event evolution

Hierarchy $\equiv$ master equation

$\Rightarrow$ without b-dependence, equivalent to a Langevin equation

$$\partial_Y T(k,Y) = \bar{\alpha} K_{\text{BFKL}} \otimes T(k,Y) - \bar{\alpha} T^2(k,Y) + \bar{\alpha}\sqrt{\kappa\alpha_s^2 T(k,Y)}\,\nu(k,Y)$$

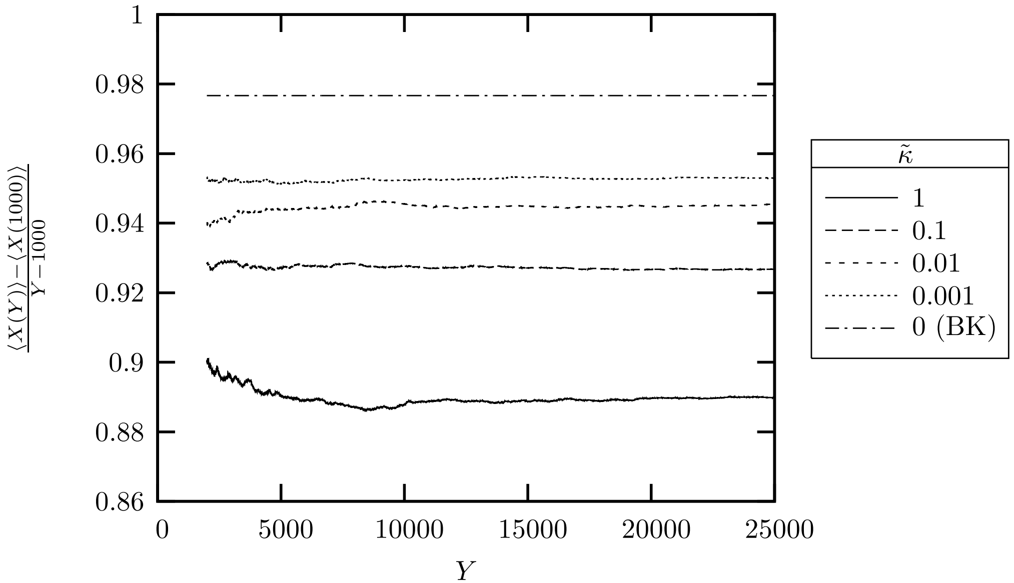
with

$$\langle \nu(k,Y)\rangle = 0 \qquad \langle \nu(k,Y)\nu(k',Y')\rangle = \frac{1}{\bar{\alpha}}\delta(Y-Y')\,k\delta(k-k')$$

Note: diffusive approximation $\rightarrow$ stochastic F-KPP equation

$$\partial_t u(x,t) = \partial_x^2 u + u - u^2 + \sqrt{2\kappa u}\,\nu(x,t)$$

[G.S., 05]

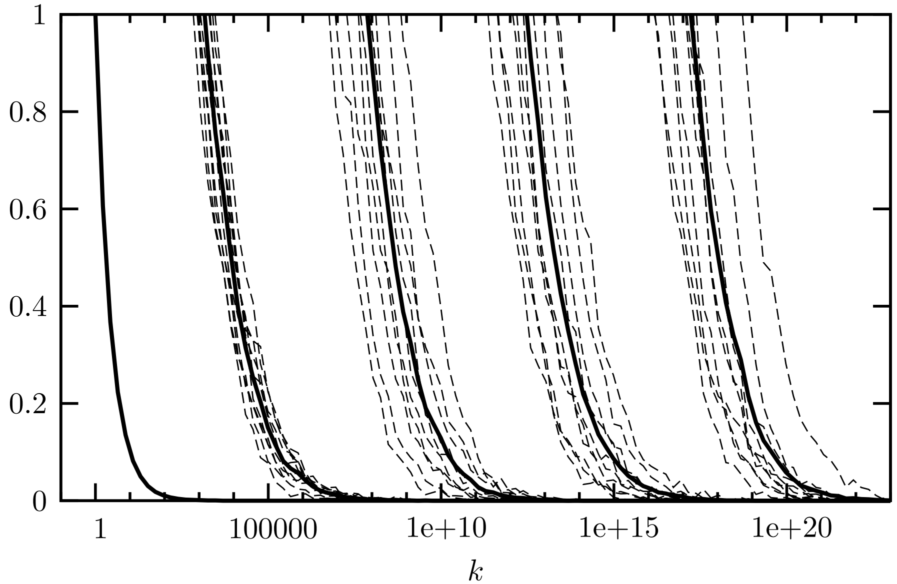


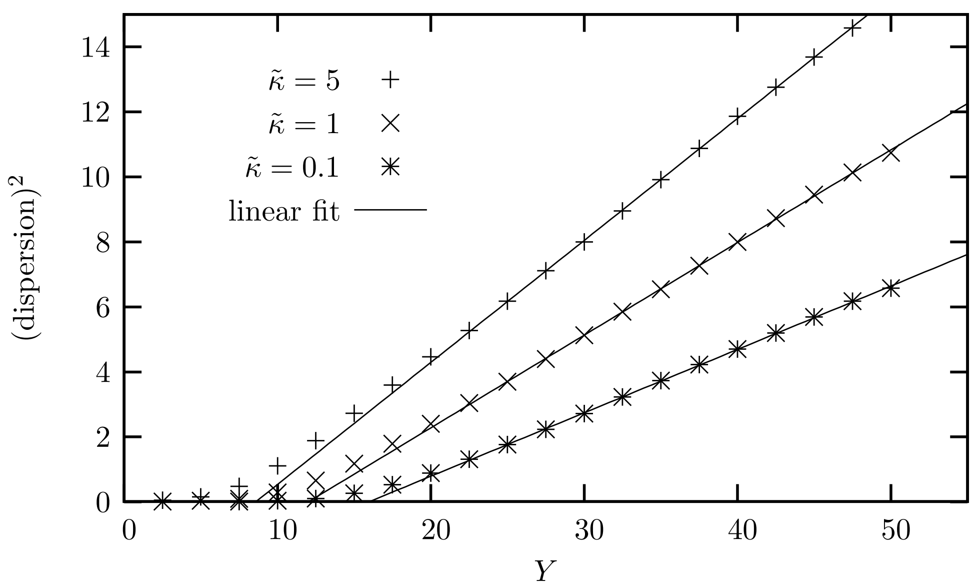
Decrease of the asymptotic velocity

For asymptotically small α_s (not true here)

$$v^* \underset{\alpha_s^2\kappa\to 0}{\longrightarrow} v_c - \frac{\bar{\alpha}\pi^2\gamma_c\chi''(\gamma_c)}{2\log^2(\alpha_s^2\kappa)} + \dots$$

[G.S., 05]





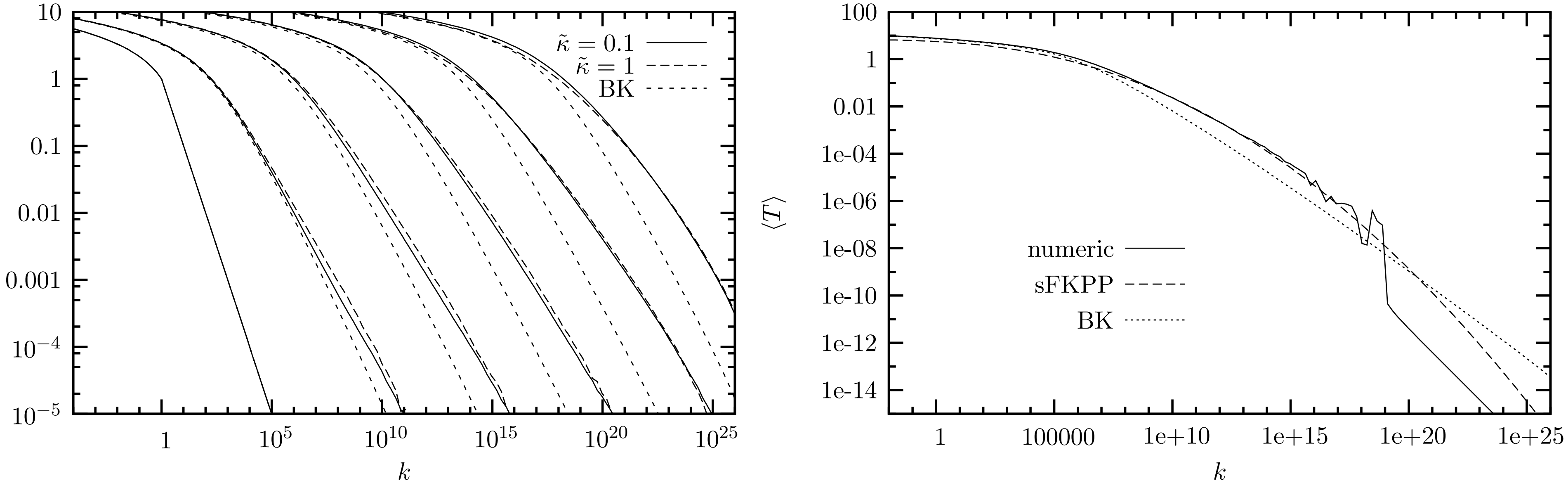
- Dispersion of the events

$$\Delta \log[Q_s^2(Y)] \approx \sqrt{D_{\text{diff}}\bar{\alpha}Y} \quad \text{with} \quad D_{\text{diff}} \underset{\alpha_s^2\kappa\to 0}{\sim} \frac{1}{|\log^3(\alpha_s^2\kappa)|}.$$

- No important dispersion in early stages of the evolution !

Averaged amplitude

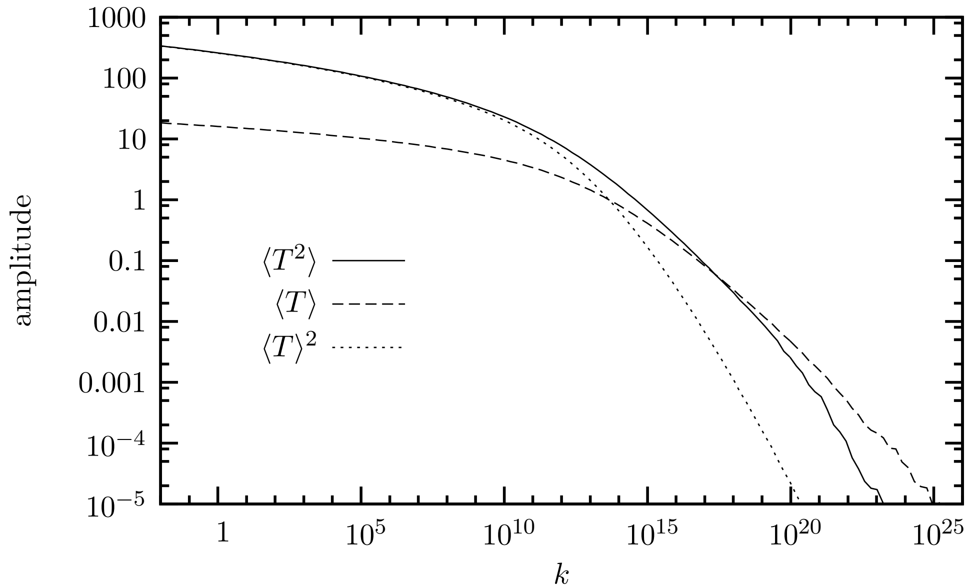
[G.S., 05]



- Clear effect of fluctuations
- Violations of geometric scaling (not in early stages)
- Agrees with predictions

Correlators

[G.S., 05]



- Dense regime: $\langle T^2\rangle \approx \langle T\rangle^2$
- Dilute regime: $\langle T^2\rangle \approx \langle T\rangle$ (pre-asymptotics!)

Conclusion

- Effects of saturation
  - Evolution equations for high-energy QCD
    Balitsky, Balitsky-Kovchegov (dipole), JIMWLK (CGC)
  - Good knowledge of the asymptotic solutions
    Traveling waves $\rightarrow$ geometric scaling, saturation scale $\propto \exp(\bar{\alpha} v_c Y)$
- Effects of fluctuations
  - Known at large-N_c
  - Consequences on saturation (*e.g.* geometric scaling violations)
  - analytical solutions: $\alpha_s \lll 1$
    numerical solutions: coherent with statistical-physics analog

Perspectives

- phenomenological tests:
  - numerical test: BK vs. data
  - do we observe geometric scaling at nonzero momentum transfer ?

Perspectives

- phenomenological tests:
  - numerical test: BK vs. data
  - do we observe geometric scaling at nonzero momentum transfer ?
- theoretical questions:
  - fluctuations
    - importance of geometric scaling violations (if relevant !)
    - analytical predictions (pomeron loops, triple pomeron vertex)
    - numerical simumations: include impact parameter
    - beyond large-N_c
  - odderon corrections