QCD at high-energy
Saturation and fluctuation effects

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Outline

Lecture 1: Evolution towards high-energy
- Motivation
- Leading log approximation: BFKL equation
- Saturation effects: BK equation
- Fluctuations

Lecture 2: Properties of the scattering amplitudes
- BK, statistical physics and geometric scaling
- Fluctuations, reaction-diffusion and diffusive scaling
- Applications
Evolution towards high-energy
Motivation

Energy $W^2 = \frac{Q^2}{x}$

Rapidity $Y = \log(1/x)$
Motivation

\[ \log(1/x) \]

\[ \log(Q^2) \]

Non perturbative

DGLAP

Q^2

x

G. Soyez
Zakopane, May 27-June 05 2006
QCD at high-energy – p. 4/64
Motivation

\[ \text{Non perturbative} \quad \text{BFKL} \quad \text{DGLAP} \]

\[ \log\left(\frac{1}{x}\right) \quad \log(Q^2) \]
Motivation

\[ Q^2 = Q_s(Y) \]

\[ \log(1/x) \]

\[ \log(Q^2) \]

Non perturbative

BFKL

DGLAP
Motivation

How to describe this in QCD?

\[ Q = Q_s(Y) \]

\[ \log(\frac{1}{x}) \]

\[ \log(Q^2) \]

QCD at high-energy – p. 4/64
Rough estimates:

**Saturation scale:** $\leftrightarrow$ partons are covering the proton

$$x g(x, Q^2) \propto Q^{-2} = \pi R_p^2$$

The gluon distribution behaves like: $x g(x, Q^2) \propto x^{-\lambda}$

$$\Rightarrow Q_s^2(Y) \propto R_p^{-2} x^{-\lambda}$$

$$R_p \approx 1 \text{ fm} \approx 5 \text{ GeV}^{-1}$$

$$Q_s^2 \approx 1 \text{ GeV}^2 \quad \text{at } x = 10^{-4}$$
Soft gluon emission

Need for some careful treatment because of Bremsstrahlung:

\[ p \]

\[ k_z = xp \]

\[ x \ll 1 \]

Probability of emission

\[ dP \sim \alpha_s \frac{dk^2}{k^2} \frac{dx}{x} \]

In the small-\( x \) limit

\[ \int_{x}^{1} \frac{dx_1}{x_1} \sim \alpha_s \log(1/x) \]
Soft gluon emission

Need for some careful treatment because of Bremsstrahlung:

\[ p \]

\[ k_z = xp \]

\[ x \ll 1 \]

\[ k_{z_1} = x_1 p \]

\[ k_z = xp \]

\[ x \ll x_1 \ll 1 \]

\[ p \]

\[ x_n \ll x_{n-1} \]

\[ x \ll x_n \]

\[ n\text{-}gluon emission: \]

\[ \int_1^x \frac{d x_n}{x_n} \int_1^{x_{n-1}} \frac{d x_{n-1}}{x_{n-1}} \ldots \int_1^{x_1} \frac{d x_1}{x_1} \sim \frac{1}{n!} \alpha_s^n \log^n (1/x) \]

\[ \rightarrow \text{At small } x: \text{ need to be resummed:} \]

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_s^n \log^n (1/x) \approx e^{\omega Y} \]
Consider a fast-moving $q\bar{q}$ dipole

Rapidity: $Y = \log(s)$
Consider a fast-moving $q\bar{q}$ dipole

Rapidity: $Y = \log(s)$

Probability $\bar{\alpha}K$ of emission
Consider a fast-moving $q\bar{q}$ dipole

Rapidity: $Y = \log(s)$

- Probability $\bar{\alpha}K$ of emission
- Independent emissions
Consider a fast-moving $q\bar{q}$ dipole. Rapidity: $Y = \log(s)$

- Probability $\bar{\alpha}K$ of emission
- Independent emissions
Consider a fast-moving $q\bar{q}$ dipole

Rapidity: $Y = \log(s)$

$n(r, Y)$ dipoles of size $r$

- Probability $\bar{\alpha}K$ of emission
- Independent emissions
- Large-$N_c$ approximation
Dipole splitting

\[ \equiv + + + \equiv \]

\[ = \frac{x - z}{(x - z)^2} - \frac{y - z}{(y - z)^2} \] ^2

\[ = \frac{\bar{\alpha}}{2\pi} \frac{(x - y)^2}{(x - z)^2(z - y)^2} \]
How to observe this system?

Scattering amplitude

\[ T(r, Y) \approx \alpha_s^2 n(r, Y) \]

Count the number of dipoles of a given size
Consider a small increase in rapidity

\[ y \]

\[ x \]
Consider a small increase in rapidity $\Rightarrow$ splitting

\[
\partial_y T(x, y; Y)
\]

\[
T(x, z; Y)
\]
Consider a small increase in rapidity $\Rightarrow$ splitting

\[
\partial_y T(x, y; Y) = T(x, z; Y) + T(z, y; Y)
\]
Consider a small increase in rapidity ⇒ splitting

\[ \partial_Y T(x, y; Y) \]

\[ T(x, z; Y) + T(z, y; Y) - T(x, y; Y) \]
Consider a small increase in rapidity ⇒ splitting

\[ \partial_Y T(x, y; Y) \]

\[ = \bar{\alpha} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ T(x, z; Y) + T(z, y; Y) - T(x, y; Y) \right] \]

[Balitsky, Fadin, Kuraev, Lipatov, 78]
BFKL evolution (3/4)

Solution  
Use Mellin space

\[ T(x, y) = T(|x - y|) = T(r) = r^{2\gamma} \quad \Rightarrow \quad \partial_Y T(r) = \bar{\alpha} \chi(\gamma)T(r) \]

\( \chi(\gamma) \) is the BFKL eigenvalues:

\[ \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) \]

\( b \)-independent solution:

\[ T(r) = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2i\pi} T_0(\gamma) \exp \left[ \bar{\alpha} \chi(\gamma)Y - \gamma \log \left( \frac{r_0^2}{r^2} \right) \right] \]
Solution in the saddle point approximation

\[ T(r, Y) \approx \frac{1}{\sqrt{Y}} \frac{r}{r_0} e^{\omega Y} \exp \left[ -\frac{\log^2(r^2/r_0^2)}{2\bar{\alpha}\chi''(1/2)Y} \right] \]

with \( \omega = 4\bar{\alpha} \log(2) \approx 0.5 \)

- Fast growth of the amplitude
- Intercept value too large
- Same with \( r_0/r \to k/k_0 \)
- Problem of diffusion in the infrared
- Violation of the Froissart bound:
  \[ T(Y) \leq C \log^2(s) \quad T(r, b) \leq 1 \]
Let us reconsider one step of the evolution

Rapidity increase $\Rightarrow$ Splitting into 2 dipoles

$$\partial_Y \langle T_{xy} \rangle = \bar{\alpha} \int_z M_{xyz} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle \right]$$

Linear BFKL

Solution: $e^{\omega_Y}$ but violates unitarity

[Balitsky, Fadin, Kuraev, Lipatov, 78]
Let us reconsider one step of the evolution

Rapidity increase $\Rightarrow$ Splitting into 2 dipoles

$\partial_Y \langle T_{xy} \rangle = \bar{\alpha} \int_z M_{xyz} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} T_{zy} \rangle \right]$

[Linear BFKL]

[Unitarity]

[Balitsky 96]
Saturation effects: BK evolution

Proportional to $T^2$
important when $T \approx 1$

- $\langle T \rangle, \langle T^2 \rangle, \ldots$: JIMWLK/Balitsky equations (at large $N_c$)
- Mean-field approximation: $\langle T^2 \rangle = \langle T \rangle^2$ (BK equation)

$$\partial_Y \langle T_{xy} \rangle = \tilde{\alpha} \int z \mathcal{M}_{xyz} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle \right]$$

Linear BFKL
Unitarity

[Kovchegov 99]

Most simple perturbative evolution equation including BFKL + saturation
Improvements due to this new term:

\[ 0 \leq T(x, y) \leq 1 \Rightarrow \text{unitarity preserved} \]
Saturation effects: pros

Improvements due to this new term:

- $0 \leq T(x, y) \leq 1 \Rightarrow$ unitarity preserved
- Cut the diffusion to the infrared

unintegrated gluon distribution
Equivalence with “usual” Feynman graphs:
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BFKL ladder
Equivalence with “usual” Feynman graphs:

BFKL ladder

fan diagram
Consider evolution of $\langle T^{(2)} \rangle$ 

\[ \partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle \]

Usual BFKL ladder

[E. Iancu, D. Triantafyllopoulos, 05] 
Also A. Mueller, S. Munier, A. Shoshi, S. Wong
Consider evolution of $\langle T^{(2)} \rangle$

$\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$

Also $\langle T^{(3)} \rangle$

[Fluctuations]

E. Iancu, D. Triantafyllopoulos, 05

Also A. Mueller, S. Munier, A. Shoshi, S. Wong
Consider evolution of $\langle T^{(2)} \rangle$

[E. Iancu, D. Triantafyllopoulos, 05]
Also A. Mueller, S. Munier, A. Shoshi, S. Wong

- Usual BFKL ladder
- Fan diagram $\rightarrow$ saturation effects
- Splitting $\rightarrow$ fluctuations, pomeron loops

\[ \partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle \]
\[ \partial_Y \langle T^{(2)} \rangle \propto \langle T^{(3)} \rangle \]
\[ \partial_Y \langle T^{(2)} \rangle \propto \langle T \rangle \]
Why fluctuations

Why do we need fluctuations?

- The fluctuation term acts as a seed for $\langle T^2 \rangle$
- Then, it grows like 2 pomerons!

\[ \sim e^{\omega Y} \quad \sim \alpha_s^2 e^{2\omega Y} \]
Why fluctuations

Why do we need fluctuations?

- The fluctuation term acts as a seed for $\langle T^2 \rangle$
- Then, it grows like 2 pomerons!

Comparable for rapidities

$$Y \sim \frac{1}{\omega_P} \log(1/\alpha_s^2) \sim \frac{1}{\alpha_s} \log(1/\alpha_s^2)$$

⇒ need to include everything.
Computation of the fluctuation term

Expected to be important for dilute target

⇒ Target made of dipoles

\[
\int_{uvz} \frac{\bar{\alpha}}{2\pi} \frac{(u - v)^2}{(u - z)^2(z - v)^2} \alpha_s^2 A_0(x_1 y_1 | uz) \alpha_s^2 A_0(x_2 y_2 | zv) \frac{1}{\alpha_s^2} \nabla_u^2 \nabla_v^2 \langle T_{uv} \rangle
\]

BFKL splitting

interaction (2GE)

dipole density
$\Rightarrow$ complicated hierarchy

$$\partial_Y \left\langle T^{(2)}(x_1, y_1; x_2, y_2) \right\rangle$$

$$= \frac{\bar{\alpha}}{2\pi} \int d^2z \frac{(x_2 - y_2)^2}{(x_2 - z)^2(z - y_2)^2} \left[ \left\langle T^{(2)}(x_1, y_1; x_2, z) \right\rangle + \left\langle T^{(2)}(x_1, y_1; z, y_2) \right\rangle - \left\langle T^{(2)}(x_1, y_1; x_2, y_2) \right\rangle - \left\langle T^{(3)}(x_1, y_1; x_2, z; z, y_2) \right\rangle + (1 \leftrightarrow 2) \right]$$

$$+ \frac{\bar{\alpha}}{2\pi} \left( \frac{\alpha_s}{2\pi} \right)^2 \int_{uvz} M_{uvz} A_0(x_1y_1|uz) A_0(x_2y_2|zv) \nabla_u^2 \nabla_v^2 \left\langle T^{(1)}(u, v) \right\rangle$$

- **Saturation**: important when $T^{(2)} \sim T^{(1)} \sim 1$ i.e. near unitarity
- **Fluctuations**: important when $T^{(2)} \sim \alpha_s^2 T^{(1)}$ or $T \sim \alpha_s^2$ i.e. dilute regime
Infinite hierarchy $\equiv$ Langevin equation

$$\partial_Y T_{xy} = \frac{\bar{\alpha}}{2\pi} \int_z \mathcal{M}_{xyz} [T_{xz} + T_{zy} - T_{xy} - T_{xz}T_{zy}]$$

$$+ \frac{1}{2} \frac{\bar{\alpha} \alpha_s}{2\pi} \int_{uvz} A_0(xyz|uz) \frac{|u - v|}{(u - z)^2} \sqrt{\nabla_u^2 \nabla_v^2 T_{uv}} \nu(u, v, z; Y)$$

where $\nu$ is a Gaussian white noise

$$\langle \nu(u, v, z; Y) \rangle = 0$$

$$\langle \nu(u, v, z; Y) \nu(u', v', z'; Y') \rangle = \delta(\bar{\alpha}Y - \bar{\alpha}Y') \delta^{(2)}(u - v') \delta^{(2)}(z - z') \delta^{(2)}(v - u')$$

- Hierarchy obtained by averaging events with different realization of the noise
- problem: non-local & off-diagonal noise!
Simple example of noise term: zero space dimension

\[ \partial_t u(t) = \sqrt{2\kappa u} \nu(t) \quad \text{with} \quad \langle \nu(t)\nu(t') \rangle = \delta(t - t') \]

\[
\text{discrete } t \quad \overset{\text{Itô}}{\Rightarrow} \quad u(t_j + \delta t) = u(t_j) + \delta t \sqrt{2\kappa u_j} \nu_j \quad \langle \nu_i \nu_j \rangle = \frac{1}{\delta t} \delta_{ij} \\
\Rightarrow \quad F(u_{j+1}) = F(u_j) + \delta t \sqrt{2\kappa u_j \nu_j} F'(u_j) + \delta t^2 \kappa u_j \nu_j^2 F''(u_j) \\
\Rightarrow \quad \partial_t \langle F(u) \rangle = \kappa \langle u F''(u) \rangle \\
\]

\[ F(u) = u^n \quad \Rightarrow \quad \partial_t \langle u^n \rangle = n(n - 1)\kappa \langle u^{n-1} \rangle \]

corresponding to the fluctuation term.
duality: target-projectile symmetry $\equiv$ boost invariance
Target vs. projectile

BFKL

Saturation

Fluctuations

duality: target-projectile symmetry $\equiv$ boost invariance

Projectile wavefunction evolution

BFKL & saturation from dipole splitting

fluctuations $\equiv$ gluon merging (or recombination, saturation)
Target vs. projectile

**BFKL**

proj.

**Saturation**

targ.

**Fluctuations**

\[ \text{duality: target-projectile symmetry } \equiv \text{boost invariance} \]

Projectile wavefunction evolution

- BFKL & saturation from dipole splitting
- fluctuations \( \equiv \) gluon merging (or recombination, saturation)

Target wavefunction evolution

- BFKL & fluctuations from dipole splitting
- saturation \( \equiv \) multiple scatterings
Reaction-diffusion process \[ A \xrightleftharpoons[\sigma]{\gamma} A + A \]

Master equation: \[ P_n \equiv \text{proba to have } n \text{ particles} \]

\[ \partial_t P_n = \gamma(n-1)P_{n-1} - \gamma nP_n + \sigma n(n+1)P_{n+1} - \sigma n(n-1)P_n \]

Particle densities: we observe a subset of \( k \) particles

\[ \langle n^k \rangle \equiv \sum_{N=k}^{\infty} \frac{N!}{(N-k)!} P_N \]
Reaction-diffusion

\[ A \overset{\gamma}{\rightleftharpoons} A + A \]

Master equation: \( P_n \equiv \text{proba to have } n \text{ particles} \)

\[
\partial_t P_n = \gamma (n-1)P_{n-1} - \gamma nP_n + \sigma n(n+1)P_{n+1} - \sigma n(n-1)P_n
\]

Evolution equation: \( \langle n^k \rangle \equiv \text{particle density/correlators} \)

\[
\partial_t \langle n^k \rangle = \gamma k \langle n^k \rangle + \gamma k(k-1) \langle n^{k-1} \rangle - \sigma k(k+1) \langle n^{k+1} \rangle
\]

Scattering amplitude for this system off a target

\[
\mathcal{A}(t) = \sum_{k=0}^{\infty} (-)^k \langle n^k \rangle_{t_0} \langle T^k \rangle_{t-t_0}
\]

\( t_0 \)-independent \( \Rightarrow \)

\[
\partial_t \langle T^k \rangle = \gamma \langle T^k \rangle - \gamma \langle T^{k+1} \rangle + \sigma \langle T^{k-1} \rangle
\]

BFKL \hspace{1cm} \text{sat.} \hspace{1cm} \text{fluct.
For QCD particle = (effective) dipoles

Dipole splitting $\equiv$ BFKL kernel

$\gamma \sim \tilde{\alpha} \frac{(x - y)^2}{(x - z)^2(z - y)^2}$

Effective dipole merging

$\sigma(x_1y_1, x_2y_2 \rightarrow uv)$

$\sim \tilde{\alpha}\alpha_s^2 \nabla_u^2 \nabla_v^2 \left\{ M_{uvz} \log^2 \left[ \frac{(x_1 - u)^2(y_1 - z)^2}{(x_1 - z)^2(y_1 - u)^2} \right] \log^2 \left[ \frac{(x_2 - v)^2(y_2 - z)^2}{(x_2 - z)^2(y_2 - v)^2} \right] \right\}$

Remarks:

- merging not always positive
- fluctuations = gluon-number fluctuations
- Can be obtained from projectile or target point of view
- Known at large $N_c$. 
Effective theory for High-Energy QCD:

- Theory for the **gluonic field**: Color
- Small-$x \equiv$ classical field radiated by frozen fast gluons
  Large-$x \equiv$ random distribution of color sources: Glass
- Large occupation number: Condensate

Equation for the probability distribution of the color charge $W_Y[\rho]$
fast gluons: frozen, source for slow partons

\[(D_{\mu} F^{\mu\nu})_a = \delta^{\nu+} \rho_a (x^-, x_\perp)\]

Random source: correlators computed using the probability distribution \(W_Y[\rho]\)

\[\langle A_i^i A_{a}^i \rangle = \int D\rho \ W_Y[\rho] \ A_{a}^i A_{a}^i\]

Strong field \(A \sim 1/g\) (equivalent to \(n \sim 1/\alpha_s\) or \(T \sim 1\))
Color Glass Condensate (3/4)

Evolution

\[ \partial_Y W_Y[\rho] = \frac{1}{2} \int_{xy} \frac{\delta}{\delta \rho_x^a} \chi_{xy}^{ab}[\rho] \frac{\delta}{\delta \rho_y^a} W_Y[\rho] \]

[Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner, 97-00]

Wilson Line

\[ V_x = P \exp \left[ ig \int dx^- A^+(x^-, \vec{x}_\perp) \right] \]

In the weak field limit \( \longrightarrow \) BFKL.
\[ S\text{-matrix: } \gamma^* \rightarrow q\bar{q} \rightarrow V_x^\dagger V_y \]

\[ S_Y = \int DA^+ W_Y [A] \frac{1}{N_c} \text{tr}(V_x^\dagger V_y) \]

Wilson Line

\[ V_x = P \exp \left[ ig \int dx^- A^+(x^-, x_\perp) \right] \]
Ladder-type diagrams $\Rightarrow$ BFKL equation

$$\partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int_z \frac{(x - y)^2}{(x - z)^2(z - y)^2} [\langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle]$$

unitarity violations
Unitarity corrections: Add fan diagrams $\Rightarrow$ Balitsky equation

$$\partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int_z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} T_{zy} \rangle \right]$$

- infinite hierarchy: Balitsky/JIMWLK
- for $\langle T_{xz,zy}^{(2)} \rangle = \langle T_{xz} \rangle \langle T_{zy} \rangle$: BK equation
Summary

gluon-number fluctuations: add fluctuations in the target

\[ \partial_Y \left\langle T_{x_1 y_1, x_2 y_2}^{(2)} \right\rangle_{\text{fluc}} = \frac{1}{2} \bar{\alpha} \left( \frac{\alpha_s}{2\pi} \right)^2 \int_{uvz} M_{uvz} A_0(1|uz) A_0(2|zv) \nabla_u^2 \nabla_v^2 \left\langle T_{uv} \right\rangle \]

- equivalent to a reaction-diffusion problem
- projectile-target duality
Asymptotic solutions for scattering amplitudes

- impact-parameter-independent BK
- BK at nonzero momentum transfer
- including fluctuations
\begin{align*}
\text{Momentum space:} \quad T(k) &= \frac{1}{2\pi} \int \frac{d^2r}{r^2} e^{i r \cdot k} T(r) = \int \frac{d^2r}{r^2} J_0(kr) T(r) \\
\text{BK equation} \quad \partial_Y T(k) &= \frac{\bar{\alpha}}{\pi} \int \frac{dp^2}{p^2} \left[ \frac{p^2 T(p) - k^2 T(k)}{|k^2 - p^2|} + \frac{k^2 T(k)}{\sqrt{4p^4 + k^4}} \right] - \bar{\alpha} T^2(k)
\end{align*}
\[ b\)-independent situation: momentum space \( (L = \log(k^2/k_0^2)) \]

\[ \partial_{\bar{\alpha}Y} T(k) = \chi_{\text{BFKL}}(-\partial_L)T(k) - T^2(k) \]

Diffusive approximation:

\[ \chi_{\text{BFKL}}(-\partial_L) = \chi\left(\frac{1}{2}\right) + \frac{1}{2}(\partial_L + \frac{1}{2})^2 \]

Time \( t = \bar{\alpha}Y \), Space \( x \approx \log(k^2), u \propto T \)

\[ \partial_t u(x, t) = \partial_x^2 u(x, t) + u(x, t) - u^2(x, t) \]

Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP)
Asymptotic solution: travelling wave

$$u(x, t) = u(x - v_c t)$$

Position: $X(t) = X_0 + v_c t$
Travelling waves

\[ \partial_t u = \partial_x^2 u + u = \omega(-\partial_x)u \]

with \( \omega(\gamma) = \gamma^2 + 1 \)

\[ \Rightarrow u(x, t) = \int \frac{d\gamma}{2i\pi} u_0(\gamma) \exp[\omega(\gamma)t - \gamma x] \]

The minimal speed is selected during evolution
Asymptotic behaviour (1/2)

More generally, it is true under the conditions:

- 0 is an unstable solution, 1 is a stable solution
- The initial condition is steep enough
- The linear equation admits solution of the form

\[ T_{\text{lin}} = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2i\pi} a_0(\gamma) \exp \left[ \omega(\gamma)Y - \gamma L \right] \]

\[ \Rightarrow \text{Travelling waves with critical speed} \]

\[ v_c = \min \frac{\omega(\gamma)}{\gamma} \quad \text{or} \quad v_c = \frac{\omega(\gamma_c)}{\gamma_c} = \omega'(\gamma_c) \]
More generally, it is true under the conditions:

- 0 is an unstable solution, 1 is a stable solution
- BFKL growth, BK damping
- The initial condition is steep enough
- Colour transparency
- The linear equation admits solution of the form

$$T_{\text{lin}} = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2i\pi} a_0(\gamma) \exp[\omega(\gamma)Y - \gamma L]$$

BFKL: $$\omega(\gamma) = \bar{\alpha} \chi(\gamma) = \bar{\alpha} \left[ 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) \right]$$

$$\Rightarrow$$ Travelling waves with critical speed

$$v_c = \min \frac{\omega(\gamma)}{\gamma} \quad \text{or} \quad v_c = \frac{\omega(\gamma_c)}{\gamma_c} = \omega'(\gamma_c)$$
Asymptotic behaviour (2/2)

BK equation: linear part \( \equiv \) BFKL kernel

\[
\begin{align*}
\gamma_c &= 0.6275 \\
v_c &= 4.8834\bar{\alpha}
\end{align*}
\]

Tail of the front:

\[
T(k, Y) = T \left( \frac{k^2}{Q_s^2(Y)} \right) \approx \log \left( \frac{k^2}{Q_s^2(Y)} \right) \left| \frac{k^2}{Q_s^2(Y)} \right|^{-\gamma_c} \exp \left( -\frac{\log^2(k^2/Q_s^2(Y))}{\bar{\alpha}\chi''(\gamma_c)Y} \right)
\]

Saturation Scale:

\[
\log(Q_s^2(Y)) = v_c Y - \frac{3}{2\gamma_c} \log(Y) - \frac{3}{\gamma_c^2} \sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}} \frac{1}{\sqrt{Y}}
\]

Geometric scaling
Geometric scaling

Numerical simulations: \( \bar{\alpha} = 0.2 \)
Geometric scaling

Numerical simulations:

\[ \bar{\alpha} = 0.2 \]
Geometric scaling

Observed in the HERA data for $F_2^p$

[Reference: Kwiecinski, Stasto; 01]
Another approach

Idea:
saturation cuts \( Q^2 < Q_s^2 \)
\( \Rightarrow BK \equiv BFKL + \text{boundary} \)

BFKL solution:
\[
T = \int \frac{d\gamma}{2i\pi} T_0(\gamma) \exp \left[ \bar{\alpha}\chi(\gamma)Y - \gamma \log(Q^2/\mu^2) \right]
\]

Conditions:
- Saddle point: \( \bar{\alpha}\chi'(\gamma_c)Y - \log(Q_s^2(Y)/\mu^2) = 0 \)
- Barrier: \( \bar{\alpha}\chi(\gamma_c)Y - \gamma_c \log(Q_s^2(Y)/\mu^2) = 0 \)

\[
\Rightarrow \gamma_c \chi'(\gamma_c) = \chi(\gamma) \text{ and } \log(Q_s^2/\mu^2) = \bar{\alpha}\chi'(\gamma_c)Y.
\]
Asymptotic solutions

The full BK equation
Question: do we have the same properties for the full BK equation?

\[ \partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \, M_{xyz} [\langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle] \]

Problem:

\[
\begin{align*}
\text{y} & \quad \text{-------------------}^\text{b_{zy}} \\
\text{z} & \quad \text{-------------------}^\text{b_{xz}} \quad \text{Diffusion/non-locality in } b \\
\text{x} & \quad \text{-------------------}^\text{b_{xy}}
\end{align*}
\]
New form of the BK equation

Solution: go to momentum space

\[
\tilde{T}(k, q) = \int d^2x \, d^2y \, e^{ik \cdot x} e^{i(q-k) \cdot y} \frac{T(x, y)}{(x-y)^2}
\]

new form of the BK equation

\[
\partial_Y \tilde{T}(k, q) = \frac{\bar{\alpha}}{\pi} \int \frac{d^2k'}{(k-k')^2} \left\{ \tilde{T}(k', q) - \frac{1}{4} \left[ \frac{k^2}{k'^2} + \frac{(q-k)^2}{(q-k')^2} \right] \tilde{T}(k, q) \right\}
\]

\[
- \frac{\bar{\alpha}}{2\pi} \int d^2k' \, \tilde{T}(k, k') \tilde{T}(k-k', q-k')
\]

- locality in \( q \) for the BFKL term

- Asymptotic solutions: study the linear kernel

We need a superposition of waves
More BFKL dynamics

Solutions of the full BFKL kernel:

\[ \tilde{T}_{\text{lin}}(k, q) = \int_{\frac{1}{2} + i\infty}^{\frac{1}{2} - i\infty} \frac{d\gamma}{2i\pi} e^{\bar{\alpha}\chi(\gamma)Y} f^\gamma(k, q)\phi_0(\gamma, q) \]

with

\[ f^\gamma(k, q) = \frac{\Gamma^2(\gamma)}{\Gamma^2\left(\frac{1}{2} + \gamma\right)} \frac{2}{|k|} \frac{|q|}{4k}^{2\gamma - 1} \left[ \begin{array}{c} 2F_1 \left( \gamma, \gamma; 2\gamma; \frac{q}{k} \right) \\ 2F_1 \left( \gamma, \gamma; 2\gamma; \frac{\bar{q}}{k} \right) \end{array} \right] - (\gamma \to 1 - \gamma) \rightarrow 1 \text{ when } k \gg q \]

\[ \Rightarrow \quad \tilde{T}_{\text{lin}}(k, q) = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{d\gamma}{2i\pi} \phi_0(\gamma, q) \exp \left[ \bar{\alpha}\chi(\gamma)Y - \gamma \log \left( \frac{k^2}{q^2} \right) \right] \]
⇒ geometric scaling for the full BK equation

Saturation Scale: same $Y$ dependence as previously

$$Q^2_s(Y) \sim q^2 \exp \left[ v_c Y - \frac{3}{2\gamma_c} \log(Y) - \frac{3}{\gamma_c^2} \sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}} \frac{1}{\sqrt{Y}} \right]$$

$$\sim q^2 \Omega^2_s(Y)$$

Tail of the front: same slope $\gamma_c$

$$T(k, Y) = T \left( \frac{k^2}{q^2\Omega^2_s(Y)} \right) \approx \log \left( \frac{k^2}{q^2\Omega^2_s(Y)} \right) \left| \frac{k^2}{q^2\Omega^2_s(Y)} \right|^{-\gamma_c}$$

Note: more careful treatment gives $Q^2_s = \Omega^2(Y) \max(q^2, Q^2_T)$

Predicts geometric scaling for $t$-dependent processes
Dependence on momentum transfer $k$: travelling waves

- Formation of a travelling wave at large $p$ (or $k$) $\Rightarrow$ Geometric scaling
- Cut-off effect in the infrared region
Saturation scale

- $Y$ dependence: converges to $u_c$
- $q$ dependence: scales like a constant or linearly ($Y = 25$)
Solutions

Fluctuation effects
Event evolution

no $b$-dependence + coarse-graining $\rightarrow$ Langevin equation

\[ \partial_Y T(k, Y) = \bar{\alpha} K_{BFKL} \otimes T(k, Y) - \bar{\alpha} T^2(k, Y) + \bar{\alpha} \sqrt{\kappa \alpha_s^2 T(k, Y)} \nu(k, Y) \]

with $\langle \nu(k, Y) \rangle = 0$

$\langle \nu(k, Y) \nu(k', Y') \rangle = \frac{1}{\bar{\alpha}} \delta(Y - Y') k \delta(k - k')$

Diffusive approximation

\[ \partial_t u(x, t) = \partial_x^2 u + u - u^2 + \sqrt{2\kappa u} \nu(x, t) \]

stochastic F-KPP
Numerical solution

Numerical solution of

$$\partial_Y T(k, Y) = \bar{\alpha} K_{BFKL} \otimes T(k, Y) - \bar{\alpha} T^2(k, Y) + \bar{\alpha} \sqrt{\kappa \alpha_s^2 T(k, Y)} \nu(k, Y)$$

Dealing with the noise term: $\, du = \sqrt{2 \kappa u \nu(t)} \Rightarrow \partial_t \langle F(u) \rangle = \kappa \langle u F''(u) \rangle$

Associated probability

$$\langle F(u) \rangle = \int du F(u) P(u, t) \quad \Rightarrow \quad \partial_t P(u, t) = \kappa \partial_u^2 \left[ u P(u, t) \right]$$

Including the initial condition $u(t = 0) = u_0$, we get

$$P_t(u_0 \rightarrow u) \equiv \text{probability to go from } u_0 \text{ to } u \text{ in a time } t.$$

Define the cumulative probability $F_{u_0, t}(u) = \int_{u_0}^{u} dv P_t(u_0 \rightarrow v)$. 
Numerical solution of

\[ \partial_Y T(k, Y) = \bar{\alpha} K_{BFKL} \otimes T(k, Y) - \bar{\alpha} T^2(k, Y) + \bar{\alpha} \sqrt{\kappa \alpha_s^2 T(k, Y)} \nu(k, Y) \]

Rapidity step \( \delta Y \):

- **Step 1:** Use probability: \( 0 < y < 1 \) uniform random variable

\[ T_{\text{noise}}(k, Y) = F_{T(k,Y),\delta Y}^{-1}(y) \]

- **Step 2:** Apply the remaining equation

\[ T(k, Y + \delta Y) = T_{\text{noise}}(k, Y) + \delta Y \left[ \bar{\alpha} K_{BFKL} \otimes T_{\text{noise}}(k, Y) - \bar{\alpha} T_{\text{noise}}^2(k, Y) \right] \]
Decrease of the velocity/exponent of the saturation scale

For $\alpha_s \ll 1$ (not true here) \cite{Mueller, Munier, Brunet, Derrida}, see S. Munier’s talk

\[
\begin{align*}
\nonumber \left\langle X(Y) \right\rangle - \left\langle X(1000) \right\rangle & \\
\nonumber Y & \\
\nonumber 0 & 5000 & 10000 & 15000 & 20000 & 25000
\end{align*}
\]

\[
v^* \rightarrow v_{BK} - \frac{\bar{\alpha} \pi^2 \gamma_c \chi''(\gamma_c)}{2 \log^2(\alpha_s^2 \kappa)} + \ldots
\]
Dispersion of the events

\[ \Delta \log[Q_s^2(Y)] \approx \sqrt{D_{\text{diff}}\bar{\alpha}Y} \quad \text{with} \quad D_{\text{diff}} \sim \frac{1}{|\log^3(\alpha_s^2\tilde{\kappa})|}. \]

No important dispersion in early stages of the evolution!
Averaged amplitude

- Clear effect of fluctuations
- Violations of geometric scaling (not in early stages)
- Agrees with predictions

[G.S., 05]
Dense regime: $\langle T^2 \rangle \approx \langle T \rangle^2$

Dilute regime: $\langle T^2 \rangle \approx \langle T \rangle$ (pre-asymptotics!)
\[ \log(Q^2) \]

\[ \mathcal{Q}_s(Y) \]

Saturation

BFKL

vetoed

[Mueller, Triantafyllopoulos, 02]

vetoed

[Mueller, Shoshi, 04]

Saturation

+ fluctuations
Evolution with saturation & fluctuations \equiv

- superposition of unitary front
  (with geometric scaling)
- with a dispersion
  (yielding geometric scaling violations)

\[
\langle T(r, Y) \rangle = \int d\rho_s \, T_{\text{event}}(\rho - \rho_s) \frac{1}{\sqrt{\pi \sigma}} \exp \left( -\frac{(\rho_s - \bar{\rho}_s)^2}{\sigma^2} \right)
\]

with \( \rho = \log(1/r^2), \rho_s = \log(Q_s^2) \)

\[
T_{\text{event}}(\rho - \rho_s) = \begin{cases} 
1 & r > Q_s \\
(r^2 Q_s^2)^{\gamma} & r < Q_s 
\end{cases}
\]
High-energy behaviour

$\log(\frac{r_0^2}{r^2})$

$T$

$\log(\frac{r_0^2}{r^2})$

$T$

Y not too large $\Rightarrow$ small dispersion $\Rightarrow$ $\langle T \rangle \approx T_{\text{event}} \Rightarrow$ geometric scaling

Y very high $\Rightarrow$ dominated by dispersion i.e. $\langle T \rangle \approx T_{\text{sat}}$

NB.: $\langle T^2 \rangle = \langle T \rangle$
### Strong noise limit

<table>
<thead>
<tr>
<th>Intermediate energies</th>
<th>High energies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean field (BK)</td>
<td>Fluctuations</td>
</tr>
<tr>
<td>Geometric scaling</td>
<td>Diffusive scaling</td>
</tr>
<tr>
<td>$\langle T \rangle = f \left[ \log \left( \frac{k^2}{Q_s^2} \right) \right]$</td>
<td>$\langle T \rangle = f \left[ \log \left( \frac{k^2}{Q_s^2} \right) / \sqrt{DY} \right]$</td>
</tr>
<tr>
<td>$\langle T^{(k)} \rangle = \langle T \rangle^k$</td>
<td>$\langle T^{(k)} \rangle = \langle T \rangle$</td>
</tr>
</tbody>
</table>

At high-energy, amplitudes are dominated by black spots *i.e.* rare fluctuations at saturation

- true for strong fluctuations
- asymptotically true in general
**Phase space & scaling**

Saturation:

\[
Q_s^2(\gamma) \approx \log(Q^2) \gamma
\]

- Geometric scaling
- DLA

\[
\sqrt{\gamma}
\]

\[
\text{saturation}
\]
Phase space & scaling

saturation:

\[ Q_s^2(Y) \]

saturation + fluctuations:

\[ \log(Q^2) \]

\[ Y \]

\[ \sqrt{Y} \]

\[ \text{DLA} \]

\[ \text{geom. scaling} \]

\[ \text{diffusive scaling} \]

G. Soyez
Zakopane, May 27-June 05 2006
QCD at high-energy – p. 59/64
**Description of $\gamma^* p$ data**

**Application:** description of $F_2^p$ data.

**Factorisation formula:** (see C. Marquet’s talk for more details)

$$\frac{\sigma_{L,T}}{d^2 b} = \int d^2 r \int_0^1 dz \left| \Psi_{L,T}(z, r; Q^2) \right|^2 T(r, b; Y)$$

- $\Psi \equiv$ photon wavefunction $\gamma^* \rightarrow q\bar{q}$: QED process
- $T \equiv$ scattering amplitude from high-energy QCD.

$$\int d^2 b T(r, b, Y) = 2\pi R_p^2 T(r; Y) \quad F_2 = \frac{Q^2}{4\pi\alpha_e} \left[ \sigma_{L}^p + \sigma_{T}^p \right]$$
Describing $F_2$

Saturation fit: [Iancu, Itakura, Munier]

\[
\langle T(\mathbf{r}, \mathbf{y}) \rangle = \begin{cases} 
(r^2 Q_s^2)^\gamma e^{-\frac{2 \log^2(r Q_s)}{c y}} & r < Q_s \\
1 - e^{-a - b \log^2(r Q_s)} & r > Q_s
\end{cases}
\]

Saturation+fluctuations fit: [in preparation]

\[
\langle T(\mathbf{r}, \mathbf{y}) \rangle = \int d\rho_s T(\mathbf{r}, \rho_s) \frac{1}{\sqrt{\pi} \sigma} e^{-\frac{(\rho_s - \bar{\rho}_s)^2}{\sigma^2}}
\]

\[
T(\mathbf{r}, \rho_s) = \begin{cases} 
(r^2 Q_s^2) & r < Q_s \\
1 & r > Q_s
\end{cases}
\]

\[
Q_s^2(\mathbf{y}) = \lambda \mathbf{y}, \rho_s = \log(Q_s^2)
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Describing $F_2$

Saturation fit: [Iancu, Itakura, Munier]

$$\langle T(r, Y) \rangle = (r^2 Q_s^2)^{\gamma c} e^{-\frac{2 \log^2 (r Q_s)}{C_Y}} \to r^2 Q_s^2$$

Saturation+fluctuations fit: [in preparation]

$$\langle T(r, Y) \rangle = \int d\rho_s T(r, \rho_s) \frac{1}{\sqrt{\pi} \sigma} e^{-\frac{(\rho_s-\bar{\rho}_s)^2}{\sigma^2}} \to \frac{1}{2} \text{erfc} \left( \frac{-\log(r^2 \bar{Q}_s^2)}{\sqrt{2DY}} \right)$$

$Y \to \infty \quad \text{Geometric scaling}$

$Y \to \infty \quad \text{Diffusive scaling}$
Both fits can describe the data for $x \leq 0.01$. 

Describing $F_2$
Conclusion

**Effects of saturation**

- Evolution equations for high-energy QCD
  Balitsky, Balitsky-Kovchegov (dipole), JIMWLK (CGC)
- Good knowledge of the asymptotic solutions
  Travelling waves $\rightarrow$ geometric scaling, saturation scale $\propto \exp(\bar{\alpha} v_c Y)$

**Effects of fluctuations**

- Known at large-$N_c$
- analytical solutions: $\alpha_s \ll 1$
- numerical solutions: coherent with statistical-physics analog
- Consequences on saturation (e.g. geometric scaling violations)
  black spots $\Rightarrow$ Diffusive scaling
phenomenological tests:

- do we observe geometric scaling at nonzero momentum transfer?
- predictions for LHC? diffusive scaling at high-energy?
phenomenological tests:
- do we observe geometric scaling at nonzero momentum transfer?
- predictions for LHC? diffusive scaling at high-energy?

theoretical questions:
- importance of geometric scaling violations
- analytical predictions
  - speed, dispersion coefficients
  - pomeron loops, triple pomeron vertex
- numerical simulations:
  - go beyond local-noise approximation
  - include impact parameter
- beyond large-$N_c$