Saturation in High-Energy QCD
Scaling laws and phenomenological applications

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C. Marquet, R. Peschanski, G.S., in preparation
Perturbative evolution in high-energy QCD:
- Leading log approx.: BFKL equation
- Unitarity/Saturation effects: Balitsky/JIMWLK and BK equation
- Fluctuation effects: towards a new evolution

Asymptotic solutions:
- saturation $\Rightarrow$ geometric scaling
- fluctuation $\Rightarrow$ Stochastic evolution $\Rightarrow$ Diffusive scaling

Phenomenological consequences
- Geometric scaling for $F_2$ and in vector meson production
- Diffusive scaling in DIS, diffractive DIS and forward gluon production
Motivation: why saturation?

How to describe this in QCD?
Motivation: why resummation?

Bremsstrahlung:

\[ p \]

\[ k_z = x p \]

\[ x \ll 1 \]

Probability of emission

\[ dP \sim \alpha_s \frac{d^2 k^2_\perp}{k^2} \frac{dx}{x} \]

In the small-\(x\) limit

\[ \int_x^1 \frac{dx_1}{x_1} \sim \alpha_s \log(1/x) \]
Motivation: why resummation?

Bremsstrahlung:

\[ p \]
\[ k_z = xp \]
\[ x \ll 1 \]

Probability of emission

\[ dP \sim \alpha_s \frac{dk_{1\perp}^2}{k_{1\perp}^2} \frac{dx}{x} \]

In the small-\( x \) limit

\[ \int_x^1 \frac{dx_n}{x_n} \ldots \int_{x_2}^1 \frac{dx_1}{x_1} \sim \frac{1}{n!} \alpha_s^n \log^n \left( \frac{1}{x} \right) \]

Same order when \( \alpha_s \log(1/x) \sim 1 \)
Perturbative evolution in high-energy QCD
Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

Probability $\bar{\alpha}K$ of emission

Independent emissions in coordinate space (transverse plane)
Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(s)$)

$Y_0 \ll Y_1 \ll Y_2 \ll Y_3$

- Probability $\bar{\alpha}K$ of emission
- Independent emissions in coordinate space (transverse plane)
- Large-$N_c$ approximation

$\mu(r,Y)$ dipoles of size $r$
How to observe this system?

\[ T(r, Y) \approx \alpha_s^2 n(r, Y) \]

Count the number of dipoles of a given size
Consider a small increase in rapidity ⇒ splitting

\[
\partial_Y T(x, y; Y) = \bar{\alpha} \int d^2z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ T(x, z; Y) + T(z, y; Y) - T(x, y; Y) \right]
\]

Emission proba from pQCD

[all possible interactions]

[Balitsky,Fadin,Kuraev,Lipatov,78]
The solution goes like

\[ T(Y) \sim e^{\omega Y} \quad \text{with} \quad \omega = 4\bar{\alpha} \log(2) \approx 0.5 \]

- Fast growth of the amplitude
- Intercept value too large
- Violation of the Froissart unitarity: \[ T(Y) \leq C \log^2(s) \quad T(r, b) \leq 1 \]
- Problem of diffusion in the infrared
Saturation effects

Multiple scattering

\( \star \) Proportional to \( T^2 \)

\( \star \) important when \( T \approx 1 \)

\( \langle \cdot \rangle \equiv \text{average over target field} \)

\[
\partial_Y \langle T(x, y; Y) \rangle \\
= \bar{\alpha} \int d^2z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle T(x, z; Y) \rangle + \langle T(z, y; Y) \rangle - \langle T(x, y; Y) \rangle \\
- \langle T(x, z; Y)T(z, y; Y) \rangle \right]
\]

But

\( \partial_Y \langle T(x, y; Y) \rangle \) contains a new object: \( \langle T(x, z; Y)T(z, y; Y) \rangle \)
In general: complete hierarchy [Balitsky, 96]

\[ \partial_Y \langle T^k \rangle \longrightarrow \langle T^k \rangle, \langle T^{k+1} \rangle \]

\[ \text{BFKL, saturation} \]

- Beyond large-\( N_c \): the hierarchy involves quadrupoles, sextupoles, ...
- Balitsky hierarchy \( \equiv \) JIMWKL eq. (Colour Glass Condensate formalism)

Mean field approx.: \( \langle T_{xz} T_{zy} \rangle = \langle T_{xz} \rangle \langle T_{zy} \rangle \)

\[ \partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} [\langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle] \]

[Balitsky 96,Kovchegov 99]

Simplest perturbative evolution equation satisfying unitarity constraint
Consider evolution of $\langle T^{(2)} \rangle$

\[ \partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle \]

E. Iancu, D. Triantafyllopoulos, 05
Also A. Mueller, S. Munier, A. Shoshi, S. Wong

Usual BFKL ladder
Consider evolution of $\langle T^{(2)} \rangle$

[E. Iancu, D. Triantafyllopoulos, 05]
Also A. Mueller, S. Munier, A. Shoshi, S. Wong

\[
\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle
\]

\[
\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(3)} \rangle
\]

Usual BFKL ladder

fan diagram \(\longrightarrow\) saturation effects
Consider evolution of $\langle T^{(2)} \rangle$

- Usual BFKL ladder
- Fan diagram $\rightarrow$ saturation effects
- Splitting $\rightarrow$ gluon-number fluctuations $\rightarrow$ pomeron loops

$\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$

$\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(3)} \rangle$

$\partial_Y \langle T^{(2)} \rangle \propto \langle T \rangle$
$\Rightarrow$ complicated hierarchy

$$\partial_Y \langle T^{(2)}(x_1, y_1; x_2, y_2) \rangle$$

$$= \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x_2 - y_2)^2}{(x_2 - z)^2 (z - y_2)^2} \left[ \langle T^{(2)}(x_1, y_1; x_2, z) \rangle + \langle T^{(2)}(x_1, y_1; z, y_2) \rangle ight.$$  

$$- \langle T^{(2)}(x_1, y_1; x_2, y_2) \rangle - \langle T^{(3)}(x_1, y_1; x_2, z; z, y_2) \rangle + (1 \leftrightarrow 2) \right]$$

$$+ \frac{\bar{\alpha}}{2\pi} \left( \frac{\alpha_s}{2\pi} \right)^2 \int_{uvz} M_{uvz} A_0(x_1y_1|uz)A_0(x_2y_2|zv) \nabla_u^2 \nabla_v^2 \langle T^{(1)}(u, v) \rangle$$

**Saturation**: important when $T^{(2)} \sim T^{(1)} \sim 1$ i.e. near unitarity

**Fluctuations**: important when $T^{(2)} \sim \alpha_s^2 T^{(1)}$ or $T \sim \alpha_s^2$ i.e. dilute regime

**Langevin formulation**: fluctuation = noise
Solutions

The BK equation
**Case 1**: no impact parameter dependence

\[ T_{xy} \rightarrow T \left( r = x - y, \ b = \frac{x + y}{2} \right) \rightarrow T(r) \]

Note:
- all arguments work for \( T(r) \) or its Fourier transform \( \tilde{T}(k) \)
- for \( \tilde{T} \), the non-linear term is simply \(-\tilde{T}^2(k)\)
BK equation: \[ \partial_Y T = \chi(-\partial_L)T - T^2 \]

When \( T \ll 1 \) BFKL works: \[ \partial_Y T = \chi(-\partial_L)T \]
Solution known:

\[
T(k) = \int \frac{d\gamma}{2i\pi} T_0(\gamma) \exp \left[ \chi(\gamma) \bar{\alpha}Y - \gamma L \right]
\]

\[
= \int \frac{d\gamma}{2i\pi} T_0(\gamma) \exp \left[ -\gamma \left( L - \frac{\chi(\gamma)}{\gamma} \bar{\alpha}Y \right) \right]
\]

\[ \Rightarrow \] Wave of slope \( \gamma \) travels at speed \( v = \chi(\gamma)/\gamma \)

\[ Y = Y_0 \]
Evolution mechanism

BK equation: \[ \partial_Y T = \chi(-\partial_L)T - T^2 \]

\( \Rightarrow \) Wave of slope \( \gamma \) travels at speed \( v = \chi(\gamma)/\gamma \)

\( T \)

\( Y = Y_0 \)

\( L = \log(\frac{r_0^2}{r^2}) \)

\[ \frac{\chi(\gamma)}{\gamma} \text{ min. when } \gamma = \gamma_c \]
Evolution mechanism

BK equation: \( \partial_Y T = \chi(-\partial_L)T - T^2 \)

\( \Rightarrow \) Wave of slope \( \gamma \) travels at speed \( v = \frac{\chi(\gamma)}{\gamma} \)

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\( \frac{\chi(\gamma)}{\gamma} \) min. when \( \gamma = \gamma_c \)
Evolution mechanism

BK equation: \[ \partial_Y T = \chi(-\partial_L)T - T^2 \] BFKL

⇒ Wave of slope \( \gamma \) travels at speed \( v = \chi(\gamma)/\gamma \)

\[ L = \log(\frac{r^2}{r_0^2}) \]

The minimal speed is selected during evoution
Consequence: geometric scaling \((Q_s \equiv \text{saturation scale} \equiv \text{front position})\)

\[
T(r, Y) = T[\sqrt{r}Q_s(Y)]
\]

with \(Q^2_s(Y) = v_c \bar{\alpha} Y\)

\[
rQ_s \ll 1 \Rightarrow \left[ r^2 Q^2_s(Y) \right]^{\gamma_c}
\]

slope \(\gamma_c\)

scaling window

\[
| \log(r^2 Q^2_s) | \lesssim \sqrt{\chi''(\gamma_c) \bar{\alpha} Y}
\]
Consequence: geometric scaling \((Q_s \equiv \text{saturation scale} \equiv \text{front position})\)

\[
T(r, Y) = T[rQ_s(Y)]
\]

with \(Q_s^2(Y) = v_c \bar{\alpha} Y\)

\[
rQ_s \ll 1 \Rightarrow \exp \left[ \log^2 \left( \frac{r^2 Q_s^2(Y)}{2 \chi''(\gamma_c) \bar{\alpha} Y} \right) \right]
\]

- Generic arguments: exponential rise + saturation \(\Rightarrow\) select \(\gamma_c\)
- Parameters fixed by linear kernel only
- Saturation effects even though \(T \ll 1\)
Consequence: geometric scaling \((Q_s \equiv \text{saturation scale} \equiv \text{front position})\)

\[
T(r, Y) = T[rQ_s(Y)]
\]

with \(Q_s^2(Y) = v_c\bar{\alpha}Y\)

\[
\text{slope } \gamma_c \approx \exp \left[ \frac{\log^2(r^2Q_s^2)}{2\chi''(\gamma_c)\bar{\alpha}Y} \right]
\]

\[
|\log(r^2Q_s^2)| \lesssim \sqrt{\chi''(\gamma_c)\bar{\alpha}Y}
\]

Interpretation: invariance along the saturation line
Numerical simulations:
Geometric scaling

Numerical simulations:

\[ T(k, Y) = T \left( \frac{k^2}{Q_s^2(Y)} \right) \approx \left( \frac{k^2}{Q_s^2(Y)} \right)^{-\gamma_c} \]

\[ Q_s^2(Y) \propto \exp(v_c Y) \]
Case 2: including impact parameter

Go to momentum space: use momentum transfer $q$

$$
\tilde{T}(k, q) = \int d^2x \, d^2y \, e^{ik \cdot x} e^{i(q-k) \cdot y} \frac{T(x, y)}{(x-y)^2}
$$

new form of the BK equation

$$
\partial_Y \tilde{T}(k, q) = \frac{\bar{\alpha}}{\pi} \int \frac{d^2k'}{(k-k')^2} \left\{ \tilde{T}(k', q) - \frac{1}{4} \left[ \frac{k^2}{k'^2} + \frac{(q-k)^2}{(q-k')^2} \right] \tilde{T}(k, q) \right\}
$$

$$
- \frac{\bar{\alpha}}{2\pi} \int d^2k' \, \tilde{T}(k, k') \tilde{T}(k - k', q - k')
$$

[C.Marquet, R.Peschanski, G.S., 05]
Numerical simulations

One can prove **analytically** that:

- **traveling wave at large** \( k \): BFKL \( \Rightarrow \) **same** \( \gamma_c, v_c \)
- **\( q \) dependence**: \( Q_s^2(q, Y) = \min(k_0^2, q^2) \exp(v_c Y) \)

**Predicts geometric scaling for** \( t \)-**dependent processes**
Solutions

Fluctuation effects
Event evolution

no $b$-dependence + coarse-graining (local fluctuations) $\longrightarrow$ Langevin equation

\[
\partial_Y T(k, Y) = \bar{\alpha} K_{\text{BFKL}} \otimes T(k, Y) - \bar{\alpha} T^2(k, Y) + \bar{\alpha} \sqrt{\kappa \bar{\alpha}_s^2 T(k, Y)} \nu(k, Y)
\]

with a Gaussian white noise $\langle \nu(k, Y) \rangle = 0$

\[
\langle \nu(k, Y) \nu(k', Y') \rangle = \frac{1}{\bar{\alpha}} \delta(Y - Y') k \delta(k - k')
\]

Remarks:

- noise $\equiv$ fluct. target field $\Rightarrow$ Different events $\equiv$ different target fields
- stochasticity as seen in detectors
- observables obtained by averaging over events
Numerical simulations

- Traveling wave/Geometric scaling for each event
- Dispersion of the events

\[ \Delta \log[Q_s^2(Y)] \approx \sqrt{D_{\text{diff}} \bar{\alpha} Y} \quad \text{with} \quad D_{\text{diff}} \sim \frac{1}{|\log^3(\alpha_s^2 \kappa)|}. \]
- Clear effect of fluctuations: dispersion ⇒ spreading
- Violations of geometric scaling
- Agrees with predictions from statistical mechanics (sFKPP)
Evolution with saturation & fluctuations \equiv

- superposition of unitary front (with geometric scaling)
- + dispersion of their position i.e. stochastic $Q_s$ (geom. scaling violations)

$$\langle T(r,Y) \rangle = \int d\rho_s \ T_{\text{event}}(\rho - \rho_s) \ P(\rho_s)$$

with $\rho = \log(1/r^2)$, $\rho_s = \log(Q_s^2)$
High-energy behaviour

Evolution with saturation & fluctuations ≡

- superposition of unitary front (with geometric scaling)
- + dispersion of their position i.e. stochastic $Q_s$ (geom. scaling violations)

$$
\langle T(r, Y) \rangle = \int d\rho_s \ T_\text{event}(\rho - \rho_s) \frac{1}{\sqrt{\pi} \sigma} \exp \left( -\frac{(\rho_s - \bar{\rho}_s)^2}{\sigma^2} \right)
$$

with $\rho = \log(1/r^2)$, $\rho_s = \log(Q_s^2)

$P(\rho_s)$ can be taken as Gaussian: mean $\bar{\rho}_s \sim \lambda Y$, dispersion $\sigma^2 \sim DY$

[C.Marquet, G.S., B.Xiao, 06]
High-energy behaviour

Evolution with saturation & fluctuations ≡

- superposition of unitary front (with geometric scaling)
- + dispersion of their position i.e. stochastic $Q_s$ (geom. scaling violations)

$$\langle T(r, Y) \rangle = \int d\rho_s T_{\text{event}}(\rho - \rho_s) \frac{1}{\sqrt{\pi} \sigma} \exp \left( -\frac{(\rho_s - \bar{\rho}_s)^2}{\sigma^2} \right)$$

with $\rho = \log(1/r^2)$, $\rho_s = \log(Q_s^2)$

$$T_{\text{event}}(\rho - \rho_s) = \begin{cases} 
1 & r > Q_s \quad \text{saturation} \\
(r^2 Q_s^2)^\gamma & r < Q_s \quad \text{geometric scaling}
\end{cases}$$
**High-energy behaviour**

Case 1: $Y$ not too large $\Rightarrow$ small dispersion $\Rightarrow$ Mean field picture $\langle T \rangle \approx T_{\text{event}}$ $\Rightarrow$ geometric scaling:

\[
\langle T \rangle = f \left[ \log \left( \frac{k^2}{Q_s^2} \right) \right] \\
\langle T^{(k)} \rangle = \langle T \rangle^k
\]
High-energy behavior

[E. Iancu, Y. Hatta, C. Marquet, G.S., D. Triantafyllopoulos, 06]

dispersion $\sim D Y$

Case 2: $Y$ higher energy $\Rightarrow$ dominated by dispersion $\Rightarrow T = 0$ or $T = 1$

$\Rightarrow$ diffusive scaling:

$$\langle T \rangle = f \left[ \log \left( \frac{k^2}{Q_s^2} \right) / \sqrt{D Y} \right]$$

$$\langle T^{(k)} \rangle = \langle T \rangle$$
Phenomenology

Geometric scaling in $F_2$
\[ \tau = \log(Q^2) - \lambda Y \]

\[ \lambda \approx 0.32 \]

\[ \sigma^{\gamma^*p}(Q^2, x) = \sigma(\tau) \]

\[ \tau = \log(Q^2) - \lambda \sqrt{Y} \]

\[ \sigma^{\gamma^*p}(Q^2, x) = \sigma(\tau) \]

\[ \tau = \log(Q^2) - \lambda Y \]

\[ \lambda \approx 0.32 \]
Factorisation formula:

\[ \sigma_{L,T}^{\gamma^*p} = \int d^2 r \int_0^1 dz \left| \Psi_{L,T}(z, r; Q^2) \right|^2 2\pi R_p^2 T(r; Y) \]

- \( \Psi_{L,T} \equiv \) photon wavefunction from QED
- dipole amplitude: scaling variable \( \tau = \log(r^2 Q_s^2 / 4) \)

\[
T(r; Y) = \begin{cases} 
T_0 \exp \left( \gamma_c \tau - \frac{\tau^2}{2\alpha c'' Y} \right) & \text{if } rQ_s < 2 \\
1 - \exp[-a(\tau + b)^2] & \text{if } rQ_s > 2
\end{cases}
\]

(Travelling wave)

(McLerran-Venugopalan)

\[ Q_s^2(Y) = k_0^2 \exp(\lambda Y) \Rightarrow \lambda \approx 0.25, \text{ in agreement with NLO BFKL predictions.} \]
Phenomenology

Geometric scaling in vector-meson production
Dipole description

[C. Marquet, R. Peschanski, G. S., to appear]

\[ t = -q^2 \]

Factorisation formula:

\[
A_{L,T}^{γ^*p \rightarrow Vp} = i \int d^2 r \int_0^1 dz \, \Psi_{L,T}(z, r; Q^2) \Psi_{L,T}^*(z, r, q : M_V^2) \, e^{izq \cdot r} \, \sigma_{\text{dip}}(r, q; Y)
\]

\[
\rightarrow \frac{d\sigma}{dt}, \quad \sigma_{\text{el}} \, \text{for} \, \rho, \phi, J/\psi, \text{DVCS}
\]
photon wavefunction: from QED
Vector-mesons wavefunction: Boosted-Gaussian model

dipole amplitude:

\[ \sigma_{\text{dip}}(r, q; Y) = 2\pi R_0^2 e^{-b|t|} T_{\text{IM}}(r, Q_s^2(q, Y)) \]

- Normalisation: only one slope \( b \) (no \( Q^2 \) dependence)

- \( T \)-matrix: \( t \)-dependent saturation scale from theoretical predictions:

\[ Q_s^2 = k_0^2 e^{\lambda Y} \quad \rightarrow \quad Q_s^2 = k_0^2 (1 + c|t|) e^{\lambda Y} \]

Hence:

\[ b, c \quad \rightarrow \quad \left. \frac{d\sigma}{dt}, \sigma_{\text{el}} \right|_{\rho, \phi} \quad \text{(201 data)} \]
Example: differential cross-section:

\[ \gamma^* p \rightarrow \rho p \]

\[ \gamma^* p \rightarrow \phi p \]

pred. for DVCS

\[ W = 71, Q^2 = 4 \]

\[ W = 82, Q^2 = 8 \]

\[ \frac{d\sigma}{d|t|} \]
Future phenomenology

Diffusive scaling at HERA and LHC
We have seen that, at high-energy,

$$\langle T(r, Y) \rangle = T \left( \frac{\log(r \bar{Q}_s)}{\sqrt{Y}} \right) = \frac{1}{2} \text{erfc} \left( \frac{\log^2(r^2 \bar{Q}_s^2)}{\sigma^2} \right)$$

with

$$\bar{Q}_s^2(Y) = k_0^2 e^{\lambda Y} \quad \text{and} \quad \sigma^2 = DY$$

Note: $\lambda$ and $D$ (or $Q_s$ and $\sigma^2$) taken as parameters

Consequences on

- DIS and diffractive DIS (DDIS)
- gluon/forward jet production
**Total cross-section**

\[ \sigma_{\text{DIS}} = \int dr \left| \Psi(r, Y; Q^2) \right|^2 \langle T(r, Y) \rangle \]

\[ \rightarrow \text{cst. } \sigma \Phi_1 \left( \frac{\log(r^2 Q_s^2)}{\sigma} \right) \]

**Diffractive cross-section**

\[ \sigma_{\text{DDIS}} = \int dr \left| \Psi(r, Y; Q^2) \right|^2 \langle T(r, Y) \rangle^2 + (q\bar{q}g) + \ldots \]

\[ \rightarrow \text{cst. } \sigma \Phi_2 \left( \frac{\log(r^2 Q_s^2)}{\sigma} \right) \]

\[ \Rightarrow \text{diffusive scaling for } \frac{1}{\sqrt{Y}} \sigma_{\text{DIS}} \text{ and } \frac{1}{\sqrt{Y}} \sigma_{\text{DDIS}} \]
**DIS and DDIS**

- Typical dipole scales in $|\Psi|^2 \otimes \langle T \rangle^{(1,2)}$:

  $$Q^2 = 100 Q_s^2$$

  ![Graph showing integrand versus $r Q_s$](image)

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<td>$r \sim 1/Q$</td>
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- Diffraction dominated by elastic amplitudes
Gluon production at LHC

dense-dilute scattering:
- $dA$ or $pp$ at forward rapidities
- dilute projectile $\rightarrow$ dipoles
- gluon at rapidity $\eta$

\[
\frac{d\sigma}{d\eta d^2k d^2b} = \frac{\bar{\alpha}}{k^2} \int \frac{d^2p}{(2\pi)^2} \phi(p, y_1) \Phi(k - p, y_2)
\]

Projectile unintegrated gluon density
\[
\phi(p, y_1) = \int \frac{d^2r}{2\pi} e^{ip\cdot r} n(r, y_1)
\]

Target contribution
\[
\Phi(k, y_2) = \int d^2r e^{ik\cdot r} \nabla^2_r \langle 2T(r, y_2) - T^2(r, y_2) \rangle
\]
\[ \Phi(k, Y) \to \frac{1}{\sigma} \exp \left[ \frac{\log^2(k^2/\bar{Q}_s^2)}{\sigma^2} \right] \implies \text{diffusive scaling for } \sqrt{Y} \Phi(k, Y) \]
Conclusion

Part 1: Evolution equations towards high-energy

Infinite hierarchy:

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Perspectives:

- beyond 2 gluon-exchange approximation
  ([J.T.Amaral, E.Iancu, G.S., D.Triantfyllopoulos, hep-ph/0611105])
- beyong large-$N_c$ approximation
Part 2: Solutions for scattering amplitudes

Geometric scaling

\[ T = T(rQ_s) \]
\[ Q_s = \exp(\lambda Y) \]

Diffusive scaling

\[ T = T[\log(rQ_s)/\sigma] \]
\[ Q_s = \exp(\lambda Y), \sigma^2 = DY \]

General predictions of saturation even when \( T \ll 1 \)

Note: Knowledge of preasympt.

Perspectives:

- Better analytic control of the fluctuation effects
- Include impact-parameter dependence
Part 3: Phenomenological consequences

**HERA:**
- geometric scaling for $F_2$, DVCS and VM-production
  $\Rightarrow$ indications for saturation
- diffusive scaling for $F_2$ and $F_2^D$ at higher energy

**LHC:**
- diffusive scaling predicted for dense-dilute collisions ($dA$ or forward $pp$)

**Perspectives:**
- Control of the interplay between geometric and diffusive scaling (HERA ?)
- More predictions for LHC
- Applications to dense-dense collisions
Conclusion

- generic scaling laws from high-energy QCD
- interesting links with statistical physics
- hints from HERA and TEVATRON