QCD saturation phenomenology: geometric scaling at HERA

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+ many other “external references”
**Outline**

- **Motivation**: QCD Bremsstrahlung & resummation $\rightarrow$ geometric scaling
- **Theoretical background**: perturbative evolution in high-energy QCD:
  - Dipole model and leading log approx.: BFKL equation
  - Unitarity/Saturation effects: Balitsky/JIMWLK and BK equation
- **Asymptotic solutions**: saturation $\Rightarrow$ geometric scaling
  - impact-parameter-independent BK: mechanism for geometric scaling
  - full BK equation: geometric scaling at nonzero momentum transfer
- **Phenomenological consequences** (mainly at HERA)
  - Geometric scaling for $F_2$ (different approaches)
  - Geometric scaling in vector meson production and DVCS
- **Conclusion(s) & discussion(s)**
Motivation: why saturation?

How to describe this in QCD?
Motivation: why resummation?

Bremsstrahlung:

\[ p \quad k_z = xp \]

\[ x \ll 1 \]

\[ p \quad k_{z1} = x_1 p \quad k_z = xp \]

\[ x \ll x_1 \ll 1 \]
Motivation: why resummation?

Bremsstrahlung:

\[ p \]
\[ k_z = xp \]

\[ x \ll 1 \]

Probability of emission

\[ dP \sim \alpha_s \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{dx}{x} \]

In the small-\( x \) limit

\[ \int_{x}^{1} \frac{dx_n}{x_n} \ldots \int_{x_2}^{1} \frac{dx_1}{x_1} \sim \frac{1}{n!} \alpha_s^n \log^n (1/x) \]

Same order when \( \alpha_s \log(1/x) \sim 1 \)
What are we dealing with?

Perturbative evolution in high-energy QCD
Consider a fast-moving $q\bar{q}$ dipole (Rapidity: $Y = \log(1/x)$) [Mueller,93]
Consider a fast-moving \( q\bar{q} \) dipole (Rapidity: \( Y = \log(1/x) \)) of size \( r \), with probability \( \bar{\alpha}K \) of emission. There are 2 degrees of freedom: rapidity \( Y \) (longitudinal) + 2D transverse coordinate.

Large-\( \mathcal{N}_c \) approximation:
Consider a small increase in rapidity ⇒ splitting

\[ \partial_Y T(x, y; Y) = \bar{\alpha} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ T(x, z; Y) + T(z, y; Y) - T(x, y; Y) \right] \]

Emission proba from pQCD

all possible interactions

[Balitsky,Fadin,Kuraev,Lipatov,78]
The solution goes like

\[ T(Y) \sim e^{\omega Y} \sim x^{-\omega} \quad \text{with} \quad \omega = 4\alpha \log(2) \approx 0.5 \]

- Fast growth of the amplitude
- Intercept value too large (NLO, \( E \) cons.)
- Violation of the Froissart unitarity:
  \[ T(Y) \leq C \log^2(s) \quad T(r, b) \leq 1 \]
- Problem of diffusion in the infrared
Saturation effects

Multiple scattering
  * Proportional to $T^2$
  * important when $T \approx 1$

$\langle \cdot \rangle \equiv$ average over target field

$$\partial_Y \langle T(x, y; Y) \rangle$$

$$= \tilde{\alpha} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle T(x, z; Y) \rangle + \langle T(z, y; Y) \rangle - \langle T(x, y; Y) \rangle ight.$$ 

$$\left. - \langle T(x, z; Y)\rangle \langle T(z, y; Y) \rangle \right]$$

But

$\partial_Y \langle T(x, y; Y) \rangle$ contains a new object: $\langle T(x, z; Y)\rangle \langle T(z, y; Y) \rangle$
In general: complete hierarchy \[ \partial_Y \langle T^k \rangle \to \langle T^k \rangle, \langle T^{k+1} \rangle \]

- Beyond large-\(N_c\): the hierarchy involves quadrupoles, sextupoles, ...
- Balitsky hierarchy \(\equiv\) JIMWLK eq. (Colour Glass Condensate formalism)

Mean field approx.: \(\langle T_{xz} T_{zy} \rangle = \langle T_{xz} \rangle \langle T_{zy} \rangle\)

\[ \partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2 (z - y)^2} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle \right] \]

Simplest perturbative evolution equation satisfying unitarity constraint
What are the solutions to the BK equation?

What are the consequences of saturation?
**Case 1:** BK equation with no impact parameter dependence

$$T_{xy} \rightarrow T \left( r = x - y, b = \frac{x + y}{2} \right) \rightarrow T(r)$$

Introduce $\rho = \log\left(\frac{r_0^2}{r^2}\right)$ (or $\rho = \log\left(\frac{k_{\perp}^2}{k_0^2}\right)$ in momentum space)

Note:

- all arguments work for $T(r)$ or its Fourier transform $\tilde{T}(k)$
- for $\tilde{T}$, the non-linear term in the BK equation is simply $-\tilde{T}^2(k)$
Evolution mechanism

BK equation: \( \partial_Y T = \chi(-\partial_\rho)T - T^2 \)

When \( T \ll 1 \) BFKL works: \( \partial_Y T = \chi(-\partial_\rho)T \)

Solution known:

\[
T(k) = \int \frac{d\gamma}{2i\pi} T_0(\gamma) \exp \left[ \chi(\gamma) \bar{\alpha} Y - \gamma \rho \right]
\]

\[
= \int \frac{d\gamma}{2i\pi} T_0(\gamma) \exp \left[ -\gamma \left( \rho - \frac{\chi(\gamma)}{\gamma} \bar{\alpha} Y \right) \right]
\]

⇒ Wave of \((\rho-\)slope \(\gamma\) travels at speed \(v = \chi(\gamma)/\gamma\)
Evolution mechanism

BK equation: \( \partial_Y T = \chi(-\partial_\rho) T - T^2 \)

\( \Rightarrow \) Wave of (\( \rho \)-)slope \( \gamma \) travels at speed \( v = \chi(\gamma) / \gamma \)

\( \rho = \log(r_0^2 / r^2) \)
Evolution mechanism

BK equation: \( \partial_Y T = \chi(-\partial_\rho)T - T^2 \)

⇒ Wave of \((\rho-)\)slope \(\gamma\) travels at speed \(v = \chi(\gamma)/\gamma\)

\(Y = Y_0\)

\(\rho = \log(r_0^2/r^2)\)

\[\frac{\chi(\gamma)}{\gamma}\text{ min. when } \gamma = \gamma_c\]
Evolution mechanism

BK equation: $\partial_Y T = \chi(-\partial_\rho)T - T^2$

$\Rightarrow$ Wave of ($\rho$-)slope $\gamma$ travels at speed $v = \chi(\gamma)/\gamma$

The minimal speed is selected during evolution
This is:

\[ T \propto \exp \left[ -\gamma_c (\rho - v_c \bar{\alpha} Y) \right] \]

Consequence: **geometric scaling** \((Q_s \equiv \text{saturation scale} \equiv \text{front position})\)

\[
T(r, Y) = T \left[ rQ_s(Y) \right] \quad \text{with} \quad Q_s^2(Y) = v_c \bar{\alpha} Y
\]

\[
\left. \begin{array}{c}
\frac{rQ_s}{\ll 1} = \left[ r^2 Q_s^2(Y) \right] \gamma_c \\
\exp \left[ \frac{\log^2 (r^2 Q_s^2)}{2\chi''(\gamma_c) \bar{\alpha} Y} \right]
\end{array} \right\} \text{slope } \gamma_c
\]

\[
| \log (r^2 Q_s^2) | \lesssim \sqrt{\chi''(\gamma_c) \bar{\alpha} Y}
\]
**Consequence:** geometric scaling \((Q_s \equiv 	ext{saturation scale} \equiv \text{front position})\)

\[
T(r, Y) = T[rQ_s(Y)] \\
\text{with } Q_s^2(Y) = v_c \bar{\alpha} Y
\]

\[
rQ_s < 1 \\
\exp \left[ \frac{\log^2(r^2Q_s^2)}{2\chi''(\gamma_c) \bar{\alpha} Y} \right]
\]

\[
| \log(r^2Q_s^2) | \lesssim \sqrt{\chi''(\gamma_c) \bar{\alpha} Y}
\]

- Generic arguments: exponential rise + saturation \(\Rightarrow\) select \(\gamma_c\)
- Parameters fixed by linear kernel only
- Saturation effects even though \(T \ll 1\)
- Came initially from an analogy between BK and F-KPP in stat. phys.
Consequence: geometric scaling \((Q_s \equiv \text{saturation scale} \equiv \text{front position})\)

\[
T(r, Y) = T[rQ_s(Y)]
\]

with \(Q_s^2(Y) = v_c \bar{\alpha}Y\)

\[
rQ_s \ll 1 \quad \Rightarrow \quad \exp \left[ \frac{\log^2(r^2Q_s^2)}{2\chi''(\gamma_c)\bar{\alpha}Y} \right]
\]

slopes \(\gamma_c\)

Interpretation: invariance along the saturation line

\[
|\log(r^2Q_s^2)| \lesssim \sqrt{\chi''(\gamma_c)\bar{\alpha}Y}
\]

[Iancu, Itakura, McLerran, 02]
Geometric scaling

Numerical simulations:

\[ Y = 25 \]
\[ Y = 20 \]
\[ Y = 15 \]
\[ Y = 10 \]
\[ Y = 5 \]
\[ Y = 0 \]

\[ \log(k^2) \]

-5  0  5  10  15  20  25  30  35  40

1e-05  1e-04  0.001  0.01  0.1  1  10

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High-Energy QCD at HERA – p. 15/38
Geometric scaling

Numerical simulations:

\[ T(k, Y) = T \left( \frac{k^2}{Q_s^2(Y)} \right) \approx \left( \frac{k^2}{Q_s^2(Y)} \right)^{-\gamma_c} \]

\[ Q_s^2(Y) \propto \exp(v_c Y) \]
Case 2: including impact parameter

Go to momentum space: use momentum transfer $q$

\[ \tilde{T}(k, q) = \int d^2x \, d^2y \, e^{ik \cdot x} e^{i(q-k) \cdot y} \frac{T(x, y)}{(x-y)^2} \]

\textit{new form of the BK equation}

\[
\partial_Y \tilde{T}(k, q) = \frac{\bar{\alpha}}{\pi} \int \frac{d^2k'}{(k-k')^2} \left\{ \tilde{T}(k', q) - \frac{1}{4} \left[ \frac{k^2}{k'^2} + \frac{(q-k)^2}{(q-k')^2} \right] \tilde{T}(k, q) \right\} \\
- \frac{\bar{\alpha}}{2\pi} \int d^2k' \, \tilde{T}(k, k') \tilde{T}(k - k', q - k')
\]

[C.Marquet, R.Peschanski, G.S., 05]
The full BK equation

1. Study BFKL with both $k$ and $q$ dependences
2. Look for power decreases in the tail ($k \gg q, k_0, k_0 \equiv$ target scale)

2 possible situations:

$q \gg k_0 \Rightarrow$ tail given by $e^{\bar{\alpha} \chi(\gamma) Y} (k^2 / q^2)^{-\gamma}$

$q \ll k_0 \Rightarrow$ tail given by $e^{\bar{\alpha} \chi(\gamma) Y} (k^2 / k_0^2)^{-\gamma}$

$\Rightarrow$ same selection mechanism with different reference scale:

$$T(k, q; Y) \propto \left[ \frac{k^2}{Q_s^2(q, Y)} \right]^{-\gamma_c}$$

with $Q_s^2(q, Y) = \begin{cases} k_0^2 e^{v_c Y} & \text{if } q \ll k_0 \\ q^2 e^{v_c Y} & \text{if } q \gg k_0 \end{cases}$
One can prove **analytically** that:

- **traveling wave at large** $k$: BFKL $\Rightarrow$ **same** $\gamma_c$, $v_c$

- **$q$ dependence**: $Q_s^2(q, Y) = \min(k_0^2, q^2) \exp(v_c Y)$

**Predicts geometric scaling for** $t$-**dependent processes**
Phenomenology

1. Geometric scaling in $F_2$

1.1. direct observation
\[ \sigma_{\gamma^* p}(Q^2, x) = \sigma(\tau) \]
\[ \tau = \tau(Q^2, x) = \frac{Q^2}{Q_s^2(x)} \]
Also observed in $eA$ collisions
Systemtic approach to find scaling relations?

Starting point:
Set of points: \((\tau_i = \tau(Q_i^2, Y_i), F_{2,i})\)

\(\rightarrow\) Quality factor: \(Q\)
Large when points on a curve.

Example: \(\tau = \log(Q^2) - \log(Q_s^2(Y))\) with \(\log(Q_s^2(Y)) = \lambda Y\)
Scan in \(\lambda \Rightarrow \lambda \approx 0.321\)
\[ \sigma_{\gamma^*p}(Q^2, x) = \sigma(\tau) \]
\[ \tau = \log(Q^2) - \log(Q_s^2(Y)) \]

- **BK with fixed coupling:**
  \[ \log(Q_s^2(Y)) = \lambda Y \]
  \[ \lambda \approx 0.32 \]

- **BK with running coupling:**
  \[ \log(Q_s^2(Y)) = \lambda \sqrt{Y} \]
  \[ \lambda \approx 1.62 \]
Phenomenology

1. Geometric scaling in $F_2$

1.2. QCD/saturation description
Factorisation formula:

\[ \sigma_{L,T}^{\gamma^* p} = \int d^2 r \int_0^1 dz \left| \Psi_{L,T}(z, r; Q^2) \right|^2 2\pi R_p^2 T(r; Y) \]

param for dipole amplitude: scaling variable \( \tau = \log(r^2 Q_s^2/4) \)

\[
T(r; Y) = \begin{cases} 
T_0 \exp \left( \gamma_c \tau - \frac{\tau^2}{2\alpha \chi''_c Y} \right) & \text{if } rQ_s < 2 \\
1 - \exp[-a(\tau + b)^2] & \text{if } rQ_s > 2 
\end{cases}
\] (travelling wave)

\[ Q_s^2(Y) = k_0^2 \exp(\lambda Y) \rightarrow \gamma_c, \chi''_c \] from LO BFKL, 3 parameters: \( \lambda, k_0, R_p \)

\[ \Rightarrow \lambda \approx 0.25, \text{ in agreement with NLO BFKL predictions.} \]
Phenomenology

2. Massive quarks effects
Charm enters the game

No heavy quark in the IIM model

Some other model does but

- are not fully QCD-based
- have clearly lower saturation momentum

**Aim:** Include the charm in the IIM model?

**Key issue:** alloc $\gamma_c$ to vary!

**Data:** ZEUS and H1 (5% renorm.)

**Domain:** $x \leq 0.01$, $Q^2 \leq 150$ GeV$^2$

<table>
<thead>
<tr>
<th>model</th>
<th>$\gamma_c$</th>
<th>$v_c$</th>
<th>$x_0$</th>
<th>$R_p$</th>
<th>$\chi^2/n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIM</td>
<td>0.6275</td>
<td>0.253</td>
<td>$2.67 \times 10^{-5}$</td>
<td>3.250</td>
<td>$\approx 0.9$</td>
</tr>
<tr>
<td>IIM+c</td>
<td>0.6275</td>
<td>0.195</td>
<td>$6.42 \times 10^{-7}$</td>
<td>3.654</td>
<td>1.109</td>
</tr>
<tr>
<td></td>
<td>0.7065</td>
<td>0.222</td>
<td>$1.19 \times 10^{-5}$</td>
<td>3.299</td>
<td>0.963</td>
</tr>
</tbody>
</table>

Note: $\gamma_c \approx 0.7$ is in better agreement with NLO BFKL predictions
Saturation scale $\sim 1$ GeV

Saturation important up to the limit of the geometric scaling window i.e. well outside the “saturated” region
Phenomenology

3. Diffraction and exclusive processes

3.1. Diffractive structure function
Geometric scaling in $F_2^D$

Same kind of factorisation but more contribs:

$$F_2^D = F_2^D(q\bar{q}) + F_2^D(q\bar{q}g) + \ldots$$

Basically: $F_2^D \propto T^2$

with the same $T$ as for $F_2$

[Marquet, 07]
Phenomenology

3. Diffraction and exclusive processes

3.2. Geometric scaling in vector-meson production
Dipole description

Factorisation formula:

\[
A_{L,T}^{\gamma^* p \rightarrow V p} = i \int d^2r \int_0^1 dz \ \Psi_{L,T}(z, r; Q^2) \Psi_{L,T}^*(z, r, q : M_V^2) e^{izq \cdot r} \sigma_{\text{dip}}(r, q; Y)
\]

\[
\rightarrow \frac{d\sigma}{dt}, \ \sigma_{\text{el}} \text{ for } \rho, \phi, J/\psi, \text{DVCS}
\]
**QCD predictions**

- Photon wavefunction: from QED
- Vector-mesons wavefunction: Boosted-Gaussian or Light-cone Gaussian

**Dipole amplitude:**

\[
\sigma_{\text{dip}}(r, q; Y) = 2\pi R_p^2 e^{-b|t|} T_{\text{GS}} (r, Q_s^2(q, Y))
\]

- Normalisation: only one slope \( b \) (no \( Q^2 \) dependence)
- \( T \)-matrix: \( t \)-dependent saturation scale from theoretical predictions:

\[
Q_s^2 = k_0^2 e^{\lambda Y} \quad \rightarrow \quad Q_s^2 = k_0^2 (1 + c|t|) e^{\lambda Y}
\]

Hence:

\[
\begin{align*}
b, c & \rightarrow \left[ \frac{d\sigma}{dt} \right]_{\rho, \phi, J/\Psi} \sigma_{\text{el}}
\end{align*}
\]

(269 data)
Example: differential cross-section:

\[ \gamma^* p \rightarrow \rho p \]

\[ \gamma^* p \rightarrow \phi p \]

pred. for DVCS
Conclusion

Part 1: Evolution equations towards high-energy

- The BK equation contains both BFKL exchanges and unitarity/saturation corrections
- Recently: pomeron-loops equations

Part 2: Solutions for scattering amplitudes

- BK equation $\Rightarrow$ geometric scaling
  - At fixed impact parameter
  - At nonzero momentum transfer
- Recently: pomeron-loops equations $\Rightarrow$ geometric + diffusive scaling
Part 3: Phenomenological consequences

We focused on the HERA phenomenology:

- geometric scaling for $F_2$:
  - Direct analysis of the data (Quality factor)
  - Iancu-Itakura-Munier model + new extension with charm

- geometric scaling in vector-meson production and DVCS
  $t$-dependence from pQCD instead of $b$-dependence postulated

▷ indications for saturation
▷ can help understanding for other experiments
Question 1: Is that really a saturation effect?
i.e. can we have geometric scaling from “something else” e.g. DGLAP evolution or energy-conservation in BFKL or non-perturbative effects?

- DGLAP & $E$-cons in BFKL have 2 problems:
  - They apply only to a limited phase-space at large $Q^2$
  - They come from UV physics (while $Q_s$ is a IR regulator)

- We go further than non-perturbative effects since $Q_s$ can be a perturbative scale
Question 2: What about RHIC physics?

- CGC & $Q_s$ are present in many RHIC analysis
- Studies of particle production in $dA$ collisions (e.g. KKT model)
- No combined HERA+RHIC analysis so far (under study)
Question 2: What about RHIC physics?

- CGC & $Q_s$ are present in many RHIC analysis
- Studies of particle production in $dA$ collisions (e.g. KKT model)
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Questions $\geq 3$: ...?